Problems

1 Two baby physicists argued over whose big brother runs faster. After a long dispute, accompanied by fighting and loud tantrums in the garden sandpit, they agreed on the following: Michael's big brother can cover half a femtoparsec per nanocentury and George's brother 25 picoastronomical units per microweek. Whose brother runs faster, and by how many nanopuerperia would he beat the slower one on a racetrack three light-microseconds long?

Results differing from the exact value by at most 2 % will be accepted.

2 Irene was making Kompot. When she was finished, she was left with two glasses of sugar solutions: 80 grams of 20 % solution and 20 grams of 80 % solution. She poured the solutions together and mixed them thoroughly. What is the mass concentration of the final solution now?

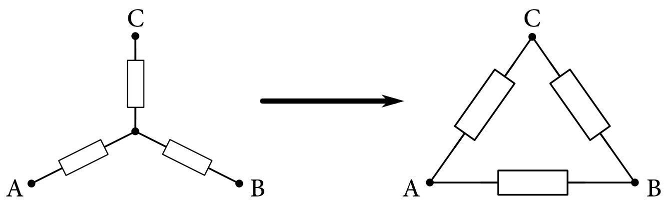
3 Peter, Paul and Arthur raced on a 100 meter track. All of them ran at a constant speed during the race. Arthur beat Paul by 20 meters. Paul beat Peter by 20 meters. How many meters did Arthur beat Peter by?

4 We put an ice cube with side length a and density ρ_i into a cylindrical glass with radius R. The ice starts to melt into water with density $\rho_w > \rho_i$. What is the maximum water level in the glass during the process of melting?

5 George went fishing. A catfish of mass m struck his fishhook. When he pulled with force F, the catfish began to move upwards with acceleration a. George, encouraged by this success, rolled up his sleeves and pulled with force $2F$. What was the acceleration of the catfish then? Assume the catfish had already given up and hung on the fishhook calmly. The density of the catfish is slightly different from the density of water. In both cases the catfish is still underwater.

Neglect the effects of hydrodynamic drag. Write the result solely in terms of quantities provided.

6 Three unknown resistors are connected in the shape of a star. If the ohmmeter is connected to two of its vertices (A–B, A–C, B–C), measured resistances are 24 Ω, 48 Ω and 56 Ω respectively. What would the measured resistances be if the resistors were connected in the shape of a triangle as pictured?



7 On September 23rd in New York, Michael the hobo slept on a bench in Central Park. It was getting cold and dark outside and he missed sunlight badly. At about midnight local time, he began contemplating how far the Sun really was and how long he would have to walk to actually see it now. The latitude of New York is 40° North. Calculate the arc distance to the nearest illuminated place on the Earth's surface.

Assume the Sun is a point light source at an infinite distance. Round the result to tens of kilometres.

8 George, having failed as a physicist, decided to become an actor. He looked forward to shooting a high speed police chase scene. He was very disappointed when he learnt the scene would actually be shot at 50 km/h, and to achieve an illusion of speed of 160 km/h on the screen, the framerate of the playback would be increased accordingly. However, the illusion was ruined as the camera also caught a flowerpot that fell out of a window in the background. What is the apparent gravity of Earth in the movie? Write the result in terms of g.

9 Matthew placed a pebble on a frictionless inclined plane. What should be the angle of the plane's inclination so that the horizontal component of the pebble's acceleration is maximal?

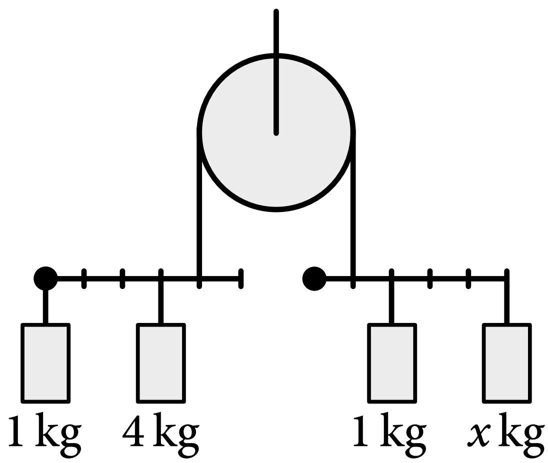
10 Matthew is a little brat. The last time he found a Y-shaped branch, he made a slingshot. The first thing he tried to hit was a sparrow, flying horizontally with constant velocity. Matthew slung a stone with velocity $v = 40$ m/s. The stone missed the sparrow very nearly, however, it hit the bird on its way down. Calculate the velocity of the sparrow, if the maximum height the stone attained was $H = 20$ m.

11 Last time Adam grew bored, he saw an empty soy milk box with a square base with side a, height b and mass m standing on a table. He started pushing the box horizontally at a certain height so that it started moving at constant speed. After a while, he realised that if he pushed high enough, the box would flip over. Calculate the maximum height where he could push the box so that it would not fall over. The coefficient of friction between the box and the table is f.

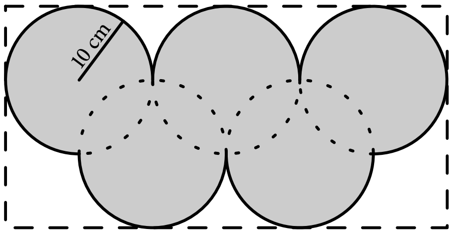
12 Lucas wandered around Norway. Exactly at the high noon on the 21st of December, he arrived at the polar circle. The Sun was not very high, but yet managed to illuminate the tip of a mountain, located 150 km to the north. How high was the mountain?

Submit the result rounded up to whole meters.

13 Find the mass x in the following system of levers and pulleys, so the system stays in equilibrium. The round spots represent pivots fixed to the wall. Note that marked segments have the same length.



14 George is looking forward to the Olympic Games in Pyeongchang, South Korea very much. He even bought a rectangular metal plate with dimensions 60 cm × 30 cm and cut out the shape of the Olympic rings from it. What is the distance between the centre of mass of his Olympic rings, and the original centre of the rectangle?



15 Gas A transforms to gas B in the chemical reaction A ↔ B. The rate of the transformation of gas A to gas B is given by

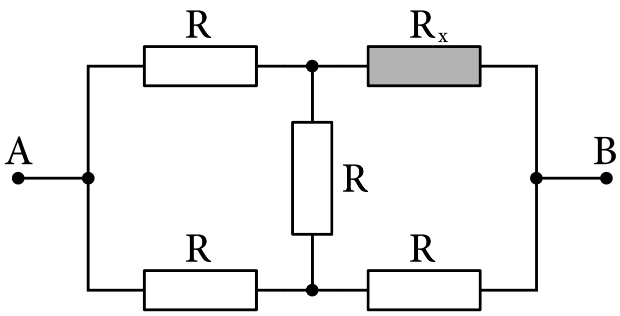
$$v = \frac{\Delta n(A)}{\Delta t} = -k_A n(A),$$

where $n(A)$ is the number of molecules of gas A in a container and k_A is the rate constant of this reaction. The rate of reaction $B \to A$ is given by an analogous equation with the rate constant k_B. We placed n_0 mol of gas A into a container and let the system reach its equilibrium between gases A and B. What is the number of molecules of gas B after the equilibrium had been achieved?

16 A red car is moving on a straight road with velocity v. A blue car is moving on another straight road, perpendicular to the first one, with velocity w. At the moment when the blue car is at the crossing, the red car is at a distance of d. Calculate the minimal distance between the cars during their journey.

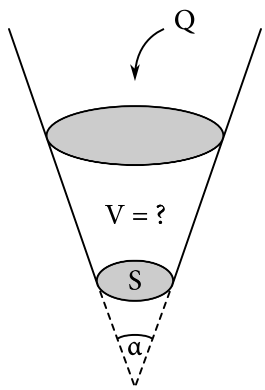
17 Francis the fireman rappels down a skyscraper. When hanging on a rope with certain length, he can use his legs to propel himself off the skyscraper to a horizontal distance of 4 m from the skyscraper wall. If he doubles the rope's length, he is able to bounce 6 m from the wall. How far would he be able to bounce, if his rope was ten times longer than the original one?

18 What are the minimal and maximal values of the resistance that can be measured between the points A and B in the provided scheme, if the resistance of resistor R_x can be chosen arbitrarily?

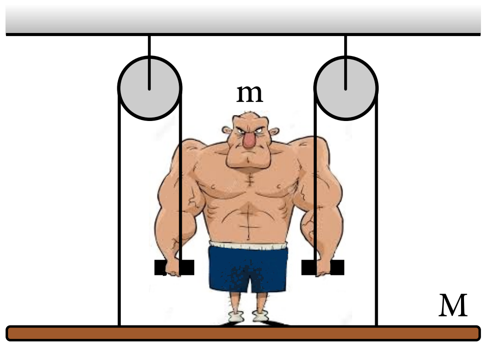


19 A funnel in the shape of a truncated cone with apex angle $\alpha = 2 \arctan \frac{1}{2}$ has a hole with area $S = \pi\ \mathrm{mm}^2$ at the bottom. We pour water in it with flow rate $Q = 2\pi$ ml/s. After some time the volume of the water in the funnel becomes stable. What is this volume?

Neglect the effects of surface tension and speed of water at its level. Round the result to whole millilitres.

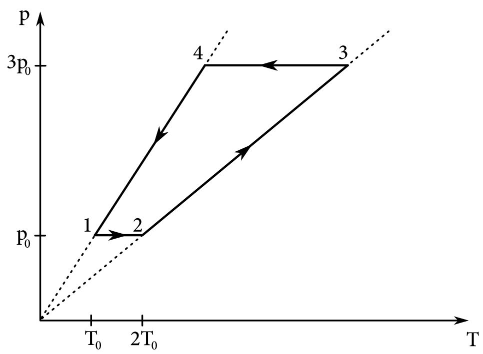


20 Hercules the strongman has mass m. His last circus stunt goes like this: he stands on a wooden board with mass $M < m$. Then he grabs two ropes going through a pair of pulleys and tied to the board and proceeds to lift himself up. What is the minimum force he has to exert to succeed?

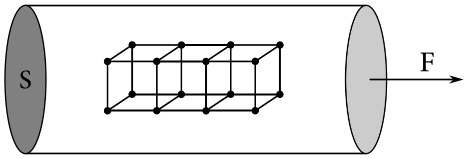


21 Consider a metal brick with proportions $a \times b \times c$. If we clasp it between two perfectly conductive plates, the following resistances can be measured: 12 Ω, 27 Ω and 75 Ω. What would be the resistance of the largest possible cube we could cut out of the brick if we place it between the same plates?

22 In the provided picture, a thermodynamic cycle with an ideal gas with an initial pressure p_0, an initial volume V_0 and an initial temperature T_0 is sketched in a pT diagram. The amount of gas in the process remains constant. Redraw the cycle in a VT diagram. Do not forget to mark all important values of volume and temperature.



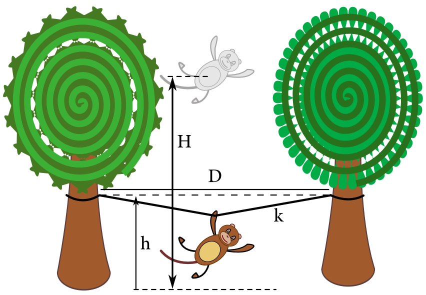
23 A metal element with density ρ, Young's modulus E and molar mass M crystalizes into a cubic lattice. We found a rod made of this metal, fastened one of its end and pulled the other end with force F along its longitudal axis. Calculate the change of the distance between atoms, if the lattice is oriented as pictured.



24 Irene is travelling by a night train. She sees the Moon through the window on her left side and a street lamp through the window on her right side. However, the inner surfaces of the windows reflect a fraction of the light, so a mirror image of the lamp is visible on her left side and a mirror image of the Moon is visible on her right side as well. No other reflections are to be considered. On the left side, the real image of the Moon is six times brighter than the lamp's reflection. On the right side, the lamp is fifty percent brighter than the reflection of the Moon.

What would the ratio of intensities of the lamp and the Moon be if there was no glass in the windows? The transmittance of glass in the train's windows is 50 %.

25 Indigenous Brazilians built a clothes line for drying their loincloths. They took two light and flexible lianas of zero rest length and of stiffness k, and tied their ends together. Then they stretched the line at a height h between two trees at a distance d. A cheeky monkey of mass M suddenly jumped off a branch at a height H above the ground and grabbed the centre of the line. How high above the ground did the monkey come to a stop before bouncing back up?



26 A massless, perfectly round clock is hung on a wall, supported by two nails, one in the middle of the clock and another on its top. To make this clock more special, we attach 12 small weights to it – at the position of k o'clock we place a weight with mass km. The upper nail can no longer withstand this tremendous load, breaks, and the clock starts turning. What is its initial angular acceleration?

27 Jacob went to an airshow. A pilot turned on speakers on the outer side of the plane before he flew above the spectators. Coincidentally, he played Jacob's favorite tune. When the plane was approaching, a tone, whose original frequency f, Jacob knew, seemed to be half an octave higher (its frequency was $\sqrt{2}$ times higher). What was the frequency Jacob heard once the plane passed him and flew straight away?

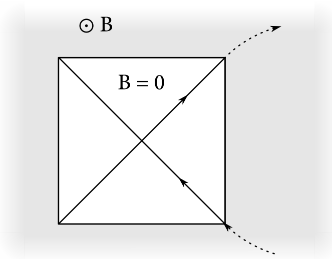
28 Joseph lay down on the bottom of a swimming pool. His eyes are now at a depth of 2 m and 2 m away from the pool's wall. Adam then stood on the edge of the pool. How tall does he seem to Joseph, if the refractive index of water is $n = 1{,}33$?

This problem does not have an analytic solution. We recommend using a calculator. The accuracy of your final and partial results should be at least two decimal places.

29 A standard 100 W light bulb emits only 4 % of its power as visible light. At night, Johny can discern such a light bulb from a distance at most 100 km. How far away would he have to fly from the Sun so that he would not be able to see it? The Sun emits 36 % of its radiation in the form of visible light.

Round the result to whole parsecs.

30 In a homogeneous magnetic field with induction B, there is a square-shaped hole, perpendicular to the fieldlines, where the field is zero. A particle with charge Q flies along the hole's diagonal, leaves the hole and returns along the hole's other diagonal. The particle passes the entire length of the diagonal in time t. What is the mass of the particle?

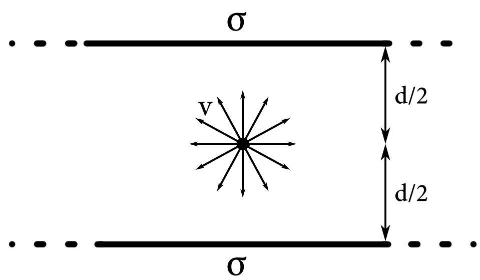


31 George has a pet flea with mass m and wants to take it for a ride on a chain carousel. Each chain has length L and is attached to the carousel structure at the distance r from the centre.

Calculate the highest angular velocity at which the flea will not fall off George's head regardless of its particular position on the head if it is capable of holding with force F.

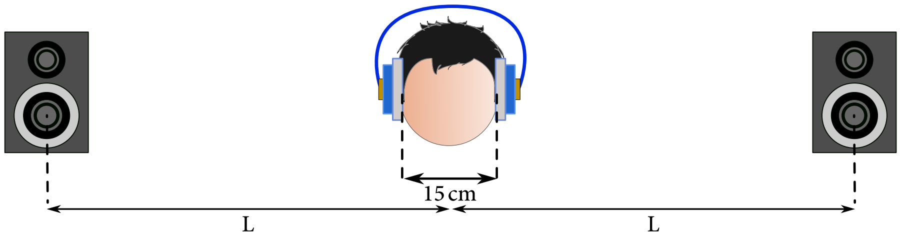
32 Rosalinda the cheeky monkey ate so many bananas that it is dangerous to be close to her. Her activity is 10^8 Bq! An average banana is 20 cm long and contains 60 μg of radioactive ^{40}K with half-life of 1,25 billion years. How many metres of banana per second should Rosalinda eat now in order to maintain her dangerous radioactivity level?

33 A proton source is placed halfway between two parallel infinite plates charged with surface density σ and separated by distance d. The source shoots protons with charge $+e$ and speed v in all directions. Which points on the plates are the protons able to reach?



34 Martin is deaf in his right ear. When he wears a pair of noise isolating earphones, he needs to boost the volume of the right earphone by 3 dB. If he switches to loudspeakers, he needs a 13 dB gain on the right speaker. Calculate the distance between the speakers if the distance between Martin's ears is 15 cm.

Round the result to centimetres. Notice that the left ear also hears the right speaker and vice-versa. Neglect the effect of the head on the sound propagation.



35 We have launched a perfectly black cube-shaped satellite into the Solar orbit. It is perfectly thermally conductive and has a reactive control system that enables it to change its attitude freely.

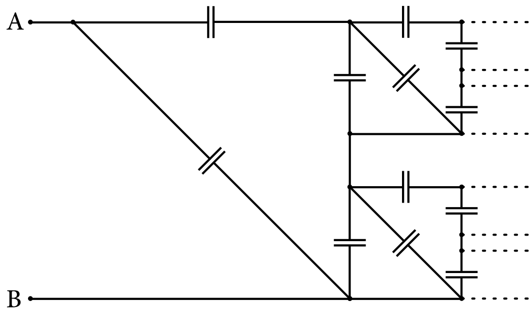
What is the ratio of the highest and lowest temperature it can attain, if solar radiation is its only energy source?

36 Clarice was fed up with power failures and decided not to depend on the commercial supplies anymore. So she bought a small nuclear reactor fusing helium nuclei. A byproduct of the fusion is beta radiation, ie. electrons leaving nuclei.

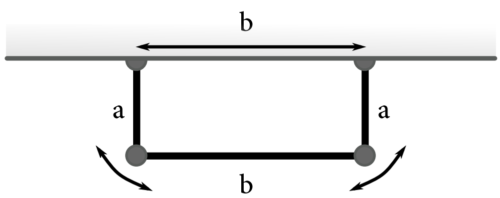
Consider a helium nucleus with radius R that emits an electron. The electron is then captured by a proton, fixed at a distance of $r = 11R$ from the helium nucleus. What must the minimum kinetic energy of the electron leaving the nucleus be, so that it is able to reach the proton?

37 Martin found an incredible number of capacitors with capacitance C in the attic of his house. The number of capacitors was practically infinite and therefore he decided to build an infinite grid of capacitors. In the first step, he took four capacitors, in the second, eight more, in the third, another sixteen, and so on.

What was the capacitance of Martin's creation when he was done?



38 A metal frame is composed of three rods, two vertical with length a and a horizontal one with length b. The joints between rods and the wall enable the rods to move freely. The linear density of the rods is λ. What is the period of small oscillations of the frame if it swings only in its original plane?



39 A planet with mass M has a single moon with mass $M/10$, placed on a circular orbit. Suddenly, Gandalf appears and magically alters the planet's mass. The moon, conserving its inertia, flies away to infinity. What is the upper bound on the planet's new mass?

40 A pirate ship has a mast with height h with a rope with length $L < h$ hanging from its top. Johny the pirate starts running with speed v on the deck. When he passes the mast, he grabs the rope and swings in a plane perpendicular to the deck passing through the mast. What is the maximal torque exerted on the foot of the mast during his motion?

You may assume Johny's speed is insufficient to bring him higher than the tip of the mast.

Solutions

1 The problem is not difficult from the physics point of view, all we need are more conventional units.

From the table of constants, we can get the value of 1 pc $\doteq 3{,}086 \times 10^{16}$ m. The femto- prefix means we multiply by 10^{-15}. Thus half of a femtoparsec is approximatelly 15,43 m.

One year has 365 days[1], each day has 24 hours, and every hour is 3600 s long. A century is 100 years. The nano- prefix means that we multiply by 10^{-9}, therefore one nanocentury equals 3,1536 s.

Astronomical unit can be found in the table of constants as 1 AU $\doteq 1{,}5 \times 10^{11}$ m. Pico- prefix is a simple multiplication by 10^{-12} so a picoastronomic unit equals 0,15 m. Each week has seven days, which is equal 604 800 s, thus one microweek is 0,6048 *slong*.

Using a calculator, we can calculate the velocity of Michael's brother as $v_\mathrm{M} \doteq 4{,}8929\,\mathrm{m\,s^{-1}}$, and George's brother as $v_\mathrm{J} \doteq 6{,}2004\,\mathrm{m\,s^{-1}}$. We see George's brother is faster.

Now let's calculate the length of the race: a light-microsecond is the distance light travels in exactly 1 μs, which is $s \doteq 3 \cdot 300\ \mathrm{m} = 900\ \mathrm{m}$. Now we can calculate the times $t_\mathrm{M} \doteq 183{,}943$ s and $t_\mathrm{J} \doteq 145{,}152$ s, and also the difference
38,791 s.

Finally, we need to express the result in nanopostpartum periods. One postpartum period is exactly 10! s = 3 628 800 s, hence a nanopostpartum period is 3,63 ms. We divide to find the final result: George's brother beats Michal's by 10 690 nanopostpartum periods.

2 The total mass of the solution is 100 g. The mass of dissolved sugar is $0{,}8 \cdot 20\ \mathrm{g} + 0{,}2 \cdot 80\ \mathrm{g} = 32\ \mathrm{g}$, thus the concentration of the mixed solution is 32%.

3 Denote speeds of Arthur, Paul and Peter v_1, v_2 and v_3 respectively. While Arthur covered an entire track, thus 100 m, Paul covered only 80 m. Therefore, the ratio of their speeds is $\frac{v_1}{v_2} = \frac{100}{80} = \frac{5}{4}$. Similarly, while Paul covered 100 m, Peter covered only 80 m, thus a speeds ratio is $\frac{v_2}{v_3} = \frac{100}{80} = \frac{5}{4}$. The ratio of Arthur's and Peter's speed is

$$\frac{v_1}{v_3} = \frac{v_1}{v_2}\frac{v_2}{v_3} = \frac{25}{16}.$$

Providing Arthur covered 100 m in time t, Peter covered only

$$v_3 t = \frac{v_3}{v_1} 100\ \mathrm{m} = \frac{16}{25} 100\ \mathrm{m} = 64\ \mathrm{m}.$$

Therefore, Arthur beat Peter by 100 m − 64 m = 36 m.

4 The cube melts and after a certain time, it starts to float. Denote an initial volume of the cube $V_c = a^3$, the volume of the unmelted remnant of the cube V_i, the volume of water V_w and the volume of the immersed part of the remnant of the cube V_p. Archimedes' principle states that the cube displaces water with volume V_p which leads to a rise of water level in the glass. Therefore, we need to calculate a volume $V = V_w + V_p$ and find its maximum as a maximum volume means maximum water level.

[1] A civil non-leap year we could also use a tropical year, the result will still fit within the precision range

The upthrust is $F_{vz} = V_p \rho_w g$. The cube floats, thus gravity and upthrust are balanced $F_g = F_{vz}$. We obtain $V_p = V_i \frac{\rho_i}{\rho_w}$. Let's express the volume V_w as a multiple of V_i. Its initial mass is $m = a^3 \rho_i$. Mass must be preserved, thus the equations $m = m_i + m_w$ must be valid throughout the entire proces of melting.

The volume of water is $V_w = \frac{m - m_i}{\rho_w}$, thus $V_w = \frac{a^3 \rho_i - V_i \rho_i}{\rho_w}$. Finally, we obtain the volume

$$V = V_w + V_p = \frac{a^3 \rho_i - V_i \rho_i}{\rho_w} + V_i \frac{\rho_i}{\rho_w} = a^3 \frac{\rho_i}{\rho_w}.$$

It implies that the water level does not depend on the amount of melted ice if the cube floats. Teoretically, if the glass was wide enough, it might happen that the ice remnant would not start floating at all. In that case, the whole cube would melt and the volume of water would be $V = a^3 \frac{\rho_i}{\rho_w}$ which is the same result as in the previous occassion. Therefore, the maximum water level is

$$h = \frac{\rho_i a^3}{\rho_w \pi R^2}.$$

$\boxed{5}$ According to Newton's Second Law of Motion, the acceleration is proportional to the acting force. What forces act on the catfish? Firstly, it is the force of George pulling the fishhook. Secondly, there is the gravitational force and the buoyant force. All the forces are constant and act only vertically, therefore they can be added simply. Let us denote their sum F'.

In such a case we can state $ma = F'$. After George pulls with force of $2F$ (ie. the pulling force gains another F), the force F' remains constant, thus $ma' = F + F'$. Excluding the F' we get the final acceleration $a' = a + \frac{F}{m}$.

$\boxed{6}$ Let's label the measured resistances as $R_{AB} = 24\ \Omega$, $R_{AC} = 48\ \Omega$ a $R_{BC} = 56\ \Omega$ and single resistors as R_1, R_2 a R_3. When measuring, there is always one resistor lying on a disconnected branch, not affecting the measured value. With this we can write:

$$R_1 + R_2 = R_{AB},$$

$$R_1 + R_3 = R_{AC},$$

$$R_2 + R_3 = R_{BC}.$$

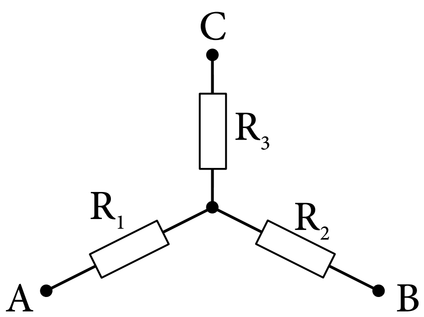


Figure 1: *Resistors connected into the shape of a star*

Single algebraic adjustments give $R_1 = 8\ \Omega$, $R_2 = 16\ \Omega$ a $R_3 = 40\ \Omega$. After reconnecting it into a triangle, we would measure a simple combination of serial and paralel architectures. Resultant resistances are:

$$R'_{AB} = \frac{R_1 (R_2 + R_3)}{R_1 + R_2 + R_3} = 7\ \Omega,$$

$$R'_{BC} = \frac{R_2 (R_1 + R_3)}{R_1 + R_2 + R_3} = 12\ \Omega,$$

$$R'_{AC} = \frac{R_3 (R_1 + R_2)}{R_1 + R_2 + R_3} = 15\ \Omega.$$

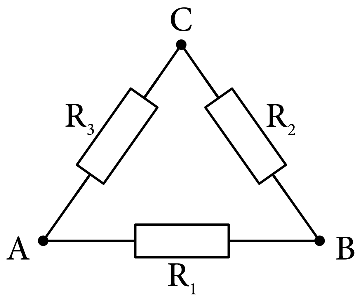
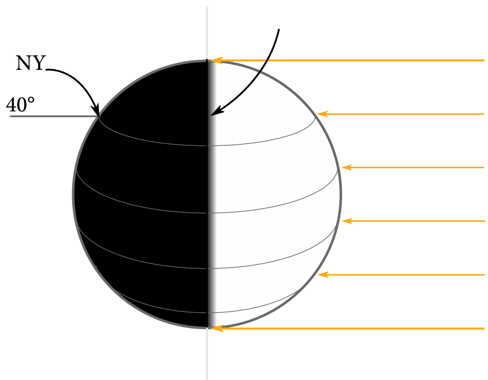


Figure 2: *Resistors connected into the shape of a triangle*

7 At midnight on an equinox, the sun is exactly on the opposite meridian to New York. The interface between the dark and light hemispheres is called the terminator.



It should be obvious that the nearest place from which we can see the Sun is the North Pole. If we walked along the terminator, we would be either getting closer or further from New York at every point except for the poles. Because our direction is not perpendicular to the line connecting New York and us at either of the poles, we know that the closest place must be one of the poles.

It is certain that the South Pole is further from New York than the North Pole, therefore it must be the North Pole. The latitude difference is then 50°. The Earth's radius is $R = 6378$ km so the arc distance is

$$d = \frac{5\pi}{18} R \doteq 5566\ \text{km} \doteq 5570\ \text{km}.$$

8 First of all, we calculate the ratio of the real framerate to the played framerate. We can do this by looking at the velocities, denote the distance the car moves between two frames d. If the frametime is τ, the velocity observed in the video is $v = \frac{d}{\tau} = df$, where $f = \frac{1}{\tau}$ is the framerate. We see that the velocity observed is proportional to the framerate. The ratio of framerates is equal to the ratio of the respective velocities $k = \frac{50\ \text{kmh}^{-1}}{160\ \text{kmh}^{-1}} = \frac{5}{16}$.

What about accelerations, how do these transform? Assume a fall from height h at constant downward acceleration. Height and time are related by $h = \frac{1}{2}gt^2$. We know the height remains constant if we change the framerate, but the time transforms to kt. Thus the expression for the observed acceleration is $h = \frac{1}{2}g'(kt)^2$ where g' is the observed gravity. To reach the result we divide these equations and see that: $g' = \frac{g}{k^2} = \left(\frac{5}{16}\right)^{-2} = 3{,}2^2 = 10{,}24g$.

9 When the rock is placed on the inclined plane, its acceleration is $g \sin\alpha$, from which the horizontal part is of magnitude $g \sin\alpha \cos\alpha = g\frac{\sin(2\alpha)}{2}$. One obtains sine maximum with 90° as argument, so the greatest horizontal acceleration is for $\alpha = 45°$.

10 As the stone hits the sparrow twice, one knows the stone and the sparrow have the same horizontal velocity. Maximum hight, which the stone reaches, determines the initial vertical velocity $v_y = \sqrt{2gH}$. The horizontal velocity can be calculated using Pythagorean theorem as $v_x = \sqrt{v^2 - v_y^2}$ which is also the velocity of the sparrow $20\sqrt{3}\,\mathrm{m\,s^{-1}} \doteq 35\,\mathrm{m\,s^{-1}}$.

11 Let us analyze the forces acting on the box when it is pushed. Firstly, there is the weight F_g, that acts at the center of gravity. Second, there is the normal reactive force N acting at the interface between box and table, the force of friction F_t, that acts against the motion and our push force F.

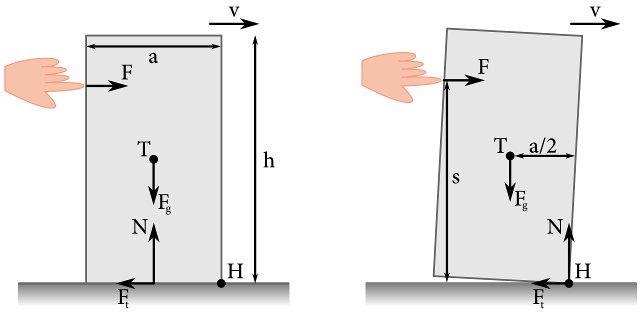


Figure 3: *A finger pushing the box*

To achieve constant speed , the horizontal forces must be equal, therefore $F = F_t$. The box does not fall through the table, so the vertical forces are equal as well, $N = F_g$. We also know $F_t = fF_g$. When the box starts turning around its edge (H), the forces of N and F_t will move to the edge. We can calculate the torque about the edge (H) efficiently, because the torques of N and F_t equal zero. Now we have only torques of F and F_g.

The box does not fall over, so the torque of F must be smaller than the torque of F_g. The position of F_g is $a/2$ (the center of gravity is $a/2$ from the edge of the box)and the position of F is s, the perpendicular distance from the table.

The box does not fall over,therefore:

$$F_g\frac{a}{2} \leq Fs,$$

$$mg\frac{a}{2} \leq fmgs,$$

$$\frac{a}{2f} \leq s,$$

from which we can calculate the maximum height where we can push of $\frac{a}{2f}$.

12 We label the latitude difference between mountain top and Lukas as φ. From this:

$$\varphi R_z = d,$$

where R_z is the Earth's radius. If the mountain height is h, trigonometry helps us (see picture below) to simply write:

$$\cos\varphi = \frac{R_z}{R_z + h}.$$

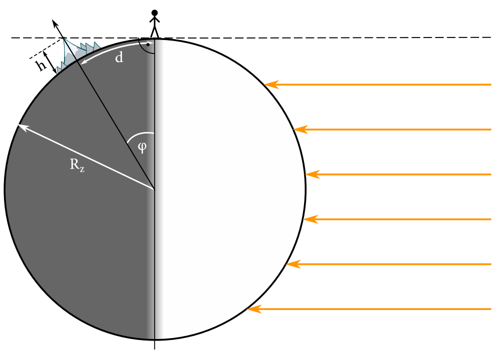


Figure 4: *Solar rays iluminating the Earth at the investigated moment*

Combining these two expressions yields the mountain height:

$$h = R_z\left(\frac{1}{\cos\left(\frac{d}{R_z}\right)} - 1\right) \doteq 1764\ \mathrm{m}.$$

13 Denote the force a rope exerts on both levers F and the gravitational acceleration g. A balance of torques on the left lever is described by the equation $1\ \mathrm{kg}\cdot g\cdot 0 + 4\ \mathrm{kg}\cdot g\cdot 3 = F\cdot 4$, thus $F = 3\ \mathrm{kg}\cdot g$. Analogously, a balance of torques on the right lever can be used to calculate the unknown mass x:

$$F\cdot 1 = 1\ \mathrm{kg}\cdot g\cdot 2 + x\cdot g\cdot 5 \Rightarrow x = 0{,}2\ \mathrm{kg}.$$

14 The olympic rings can be divided into shapes that we can describe easily. As we can see in the picture, the rings consist of rectangle 40 cm × 10 cm, five half-circles and two quarters of a circle with radius of $r = 10$ cm.

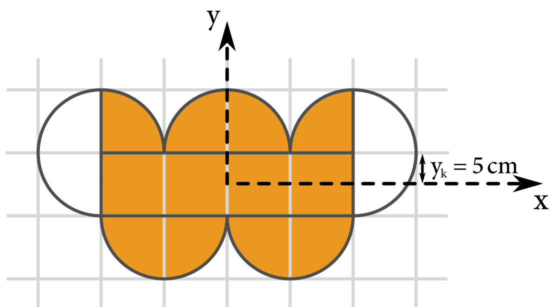


Figure 5: *Division of rings to more elementary shapes*

Let the origin of our cartesian system be in the center of the original rectangle. This will make the problem easier, because the distance of the centers will be the position vecto r of the new center of gravity. The olympic rings are symetrical according to the vertical axis, the x part of the position will be zero. We need to calculte the y part. This is done by weighted aritmetic mean:

$$y_T = \frac{\sum_i m_i y_i}{\sum_i m_i}.$$

The rings can be divided into a rectangle, three half circles and two quarter circles (marked in the picture). The sum of the y parts of these shapes is zero.The remaining half-circles' center of gravity is in $y_k = 5$ cm. Let the density of the material be σ. We now get the position of the center of gravity.

$$y_T = \frac{2\sigma \frac{\pi r^2}{2} \frac{r}{2}}{\sigma 4r^2 + \sigma 3\pi r^2} = \frac{5\pi}{3\pi + 4} \text{ cm} \doteq 1{,}17 \text{ cm}.$$

15 Balance occurs when rates of gas transformations are same. Denote balanced number of molecules of gases n_A^{eq} and n_B^{eq}. The condition of balance gives

$$k_A n_A^{eq} = k_B n_B^{eq}.$$

The total number of molecules of gases must be preserved due to stoichiometry of the reaction, thus $n_A + n_B = n_0$. Therefore, the number of molecules of gas B is $n_B = \frac{k_A}{k_A + k_B} n_0$.

16 Consider two cars moving down two mutually perpendicular roads. Denote the road of the red car x and the road of blue one y. We need to find a mutual distance which encourages us to change a frame of reference. Let's look at the situation in the frame of reference of the red car. That means the blue car moves askew with velocity $\vec{u} = \vec{v} + \vec{w}$.

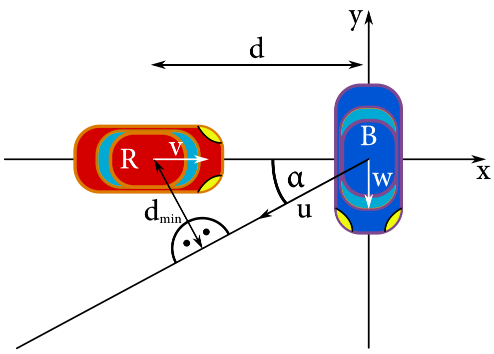


Figure 6: *Moving cars*

Now consider a moment when the blue car crosses the crossroad and mark it as initial. We find an angle between velocity $\vec{u}$ and x axis. Vector $\vec{u}$ can be decomposed as follows: $v = u \cos\alpha$ and $w = u \sin\alpha$. A quotient of these equations gives $\tan\alpha = \frac{w}{v}$.

In the red car's frame of reference, a trajectory of the blue car is deflected from x axis by an angle α. The shortest distance between cars correspond to the perpendicular line on the blue car trajectory passing the red car. We know that the red car is d away from the crossroad and we also know the deflection of the blue car trajectory. Therefore, a perpendicular distance is

$$d_{min} = d \sin\left(\arctan\left(\frac{w}{v}\right)\right) = d \frac{w}{\sqrt{v^2 + w^2}}.$$

17 Denote the lenght of the rope L and the speed the fireman can reach immediately after the bounce v. Energy must be conserved, therefore, if he bounces with maximum speed, he will always be lifted same hight.

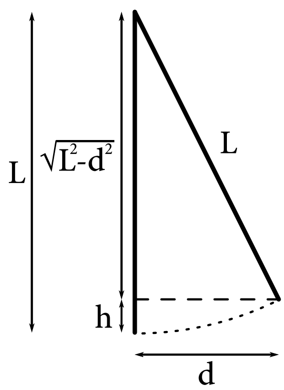


Figure 7: *Fireman on the rope*

Let's compare the hights he reached in the first and the second occassion:

$$L - \sqrt{L^2 - 4^2\ \mathrm{m}^2} = 2L - \sqrt{4L^2 - 6^2\ \mathrm{m}^2}.$$

We need to find the lenght L:

$$L - \sqrt{L^2 - 16\ \mathrm{m}^2} = 2L - \sqrt{4L^2 - 36\ \mathrm{m}^2}$$

$$\sqrt{4L^2 - 36\ \mathrm{m}^2} = L + \sqrt{L^2 - 16\ \mathrm{m}^2} \qquad \Big/ \, ()^2$$

$$L^2 - 10\ \mathrm{m}^2 = L\sqrt{L^2 - 16\ \mathrm{m}^2} \qquad \Big/ \, ()^2$$

$$L^2 = 25\ \mathrm{m}^2 \Rightarrow L = 5\ \mathrm{m}$$

The height which the fireman can be lifted to, is $h = 5\ \mathrm{m} - \sqrt{5^2\ \mathrm{m}^2 - 4^2\ \mathrm{m}^2} = 2\ \mathrm{m}$. In the last occassion, the fireman bounces to the distance d_3:

$$2\ \mathrm{m} = 50\ \mathrm{m} - \sqrt{2500\ \mathrm{m}^2 - d_3^2} \Rightarrow d_3 = 14\ \mathrm{m}.$$

18 The resistor R_x can be replaced by a resistor with an arbitrary resistance. That means we can change it continuously between values 0 and $\infty\ \Omega$. We are concerned about a range of the overall resistance which also changes continuously, therefore, we just need to find total resistances in extreme occassions.

If we put $R_x = 0\ \Omega$, the resistor behaves as an ideal conductor. The ideal conductor has zero resistance, thus its lenght can be changed arbitrarily. Therefore, we can join nodes connected with the ideal conductor to one node. The scheme can be redrawn this way:

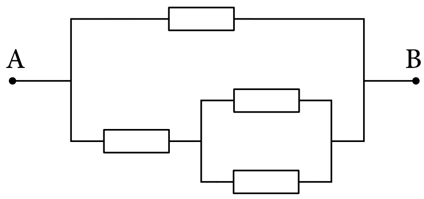


Figure 8: *The circuit with $R_x = 0\ \Omega$*

We obtain only series and parallel connection of resistors. The two parallel resistors in the bottom branch have a total resistance $\frac{R}{2}$, thus

$$R_0 = \frac{R \cdot \frac{3}{2}R}{R + \frac{3}{2}R} = \frac{3}{5}R.$$

If we put $R_x = \infty\ \Omega$, it behaves like there is an ideal insulator. Therefore, this branch can be removed from the circuit. We obtain following scheme:

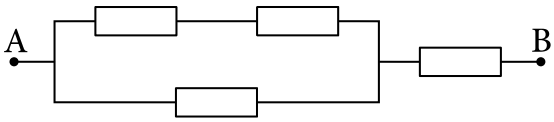


Figure 9: *The circuit with $R_x = \infty\ \Omega$*

The resistance of the parallel resistors is $\frac{2}{3}R$, thus the total resistance is

$$R_\infty = \frac{5}{3}R.$$

The overall resistance varies between $\frac{3}{5}R$ and $\frac{5}{3}R$.

Some of you may object that we do not know exact value of total resistance for general R_x. After all, the fact that a function is continuous, does not mean that it is also monotonic. Therefore, it may reach extremal values also out of limit cases. Hence, we will show briefly how total resistance of the circuit R_Σ can be calculated for general R_x.

Firstly, let's focus only on three resistors R in left part of the circuit. They are connected to the triangle. If we measured resistance between either two nodes, we would get exactly $\frac{2}{3}R$ in all three cases. The resistance of these resistors is fixed, thus we can replace them with any configuration of resistors which gives same result. Let's do it. We will replace the original configuration with three resistors with resistances $\frac{R}{3}$ connected to the star. This transformation is called triangle–star (or Δ-Y). We let you verify that these two configurations are really equivalent.

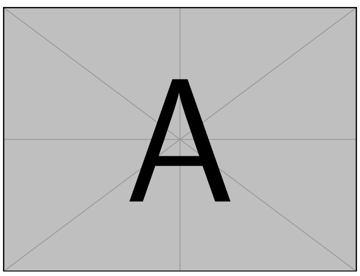


Figure 10: *Transformed scheme*

Now, a final scheme consists of resistors connected only in series and parallel, thus we can easily obtain total resistance

$$R_\Sigma(R_x) = \frac{R}{3} + \frac{\frac{4}{3}R\left(\frac{R}{3} + R_x\right)}{\frac{5}{3}R + R_x}.$$

It is really monotone function at interval $[0; \infty]$, so our original result is correct.

19 The stable level is achieved, if the flow rates of incoming and outcoming water are equal. The outcoming flow rate is from the Torricelli's equation: $Q = \sqrt{2hg}$, where h is the water level. Now we only have to calculate the volume from geometry:

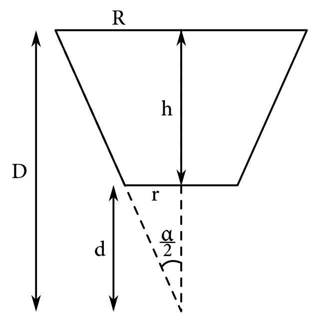


Figure 11: *Cross section of funnel*

The radius of the smaller cone is $r = 1$ mm, its heights is $d = 2$ mm. The bigger cone is higher by h: $D = d + h = 202$ mm, the big radius is $R = 101$ mm. The final volume of water is 2158 cm^3.

20 If the strogman pulls ropes with a sufficiently large force, he starts to move upwards. Since we are interested in the stable mode, we are dealing with static problem. In such problems, detailed analysis of acting forces leads to a correct result.

The strongman acts on each rope with force T. Since we consider all ropes and pulleys as ideal, force T will act with the same magnitude along whole rope length. According to Newton's third law, the ropes act upwards on the strongman with force $2T$. Besides this, there is the strongman's weight mg and normal force N emerging from the board. The board is lifted up with force $2T$ (ropes), its weight is Mg and it also feels the normal force of strongman, pointing down.

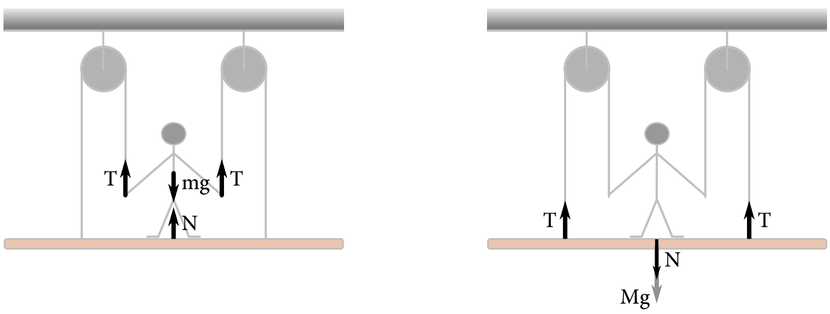


Figure 12: *Left: forces acting on the strongman. Right: Forces acting on the board*

We find the system in equilibrium, so the sum of forces has to be zero:

$$2T - mg + N = 0,$$

$$2T - Mg - N = 0.$$

Extracting N and substituting it to the other equation gives:

$$T = \frac{(m+M)\,g}{4},$$

$\boxed{21}$ Resistance of the block with resistivity ρ, lenght x and cross-section S is $R = \rho\frac{x}{S}$. It is clear that the resistance of the block is the least when the biggest faces touch conductive plates and the shortest edge determines the distance between plates. Denote the shortest edge a. It is an edge of the biggest cube we can cut from the block. Its resistance is

$$R_\mathrm{k} = \rho\frac{a}{a^2} = \frac{\rho}{a}.$$

Notice what we obtain if we multiply two largest resistances – $R_b = \rho\frac{b}{ac}$ and $R_c = \rho\frac{c}{ab}$ – and calculate its square root:

$$\sqrt{R_b R_c} = \sqrt{\rho\frac{b}{ac}\rho\frac{c}{ab}} = \rho\frac{1}{a} = \rho\frac{a}{a^2}.$$

This is exactly the resistance of the biggest cube we can cut out. Using numerical values do we obtain a numerical result $\sqrt{R_b R_c} = \sqrt{27\ \Omega \cdot 75\ \Omega} = 45\ \Omega$.

$\boxed{22}$ Before we start with redrawing the pT diagram into VT diagram, we have to find out, what thermodynamic processes, volumes and temperatures are present during each step. We will use the ideal gas equation $pV = NkT$, where N is constant number of particles.

Obviously, the first step $1 \to 2$ is an isobaric process, where temperature becomes two times higher, so volume also increases from the value $V_0 = \frac{NkT_0}{p_0}$ to $2V_0$. Second step $2 \to 3$ is due to linear relationship between pressure and temperature (ratio $\frac{p}{T} = \frac{Nk}{V}$ is constant) considered as isochoric. Pressure gets three times higher, $6T_0$ and volume stays $2V_0$. Process $3 \to 4$ isobaric and $4 \to 1$ again isochoric. With this, we can say that in state 4, the volume is V_0 and temperature $3T_0$.

The isobaric part of the VT diagram will look very similar to the isochoric in the VT diagram. Ratio $\frac{V}{T} = \frac{Nk}{p}$ stays constant, so it will be a linear function, intersecting the graph origin. The redrawn VT diagram thus looks as follows:

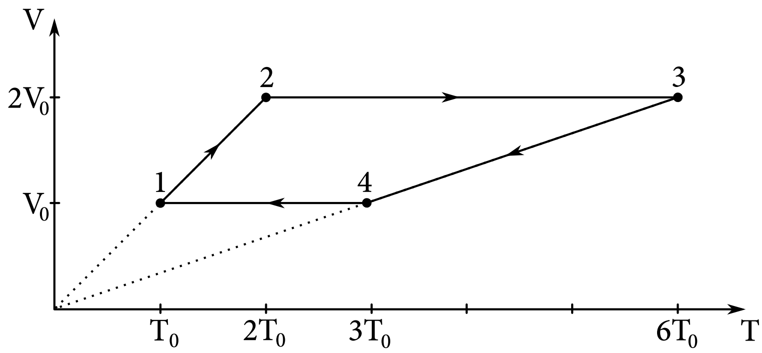


Figure 13: *Cycle redrawn to VT diagram*

23 By Hooke's law, the relative extension of the rod is $\varepsilon = \frac{F}{SE}$. The relative extension of the rod is an intensive quantity, then it is also the relative extension of interatomic distances a of the lattice. We can determine the initial value of a by considering density of the rod. The crystal in question has cubic lattice, i.e. each atom has six neighbours in a cube with edge of length a. Atomic mass of this crystal is $\frac{M}{N_A}$, where N_A is the Avogadro constant. Therefore, the material density can be expressed as

$$\rho = \frac{M}{N_A a^3} \Rightarrow a = \sqrt[3]{\frac{M}{N_A \rho}}$$

and the total extension per horizontal bond is

$$\sqrt[3]{\frac{M}{N_A \rho}} \frac{F}{SE}.$$

24 On the left side, Irene sees the Moon's brightness to be $\frac{I_\mathrm{M}}{2}$, where I_M is the true apparent brightness of then Moon, and a lamp of brightness $r\frac{I_\mathrm{L}}{2}$, where r is the unknown fraction of reflected light. Furthermore, we know that the relation between the brightnesses on the left is

$$\frac{I_\mathrm{M}}{2} = 6r\frac{I_\mathrm{L}}{2}.$$

We can perform analogous steps for the right side of the train, obtaining

$$\frac{I_\mathrm{L}}{2} = \frac{3}{2}r\frac{I_\mathrm{M}}{2}.$$

Since we are interested in the ratio $\frac{I_\mathrm{L}}{I_\mathrm{M}}$, we can just divide the upper two equations. Let's call the ratio p. Notice that the information about 50 % passed light is completely irrelevant (each photon passes through exactly one window) and also that the unknown parameter r disappears. After dividing the equations we obtain

$$p = \frac{3}{12}\frac{1}{p}.$$

After solving this, the answer to this problem takes the value

$$p = \frac{1}{2}.$$

25 We will solve this problem using the conservation of energy law. Assume, that the rope consists of two springs. The potential energy of a spring is $\frac{1}{2}k(\Delta L)^2$, where ΔL is the length by which the spring is extended. At the begining the ropes had zero length. When it is strained between the trees, both springs have length of $D/2$. Thus the initial potential energy is $\frac{1}{2}k\left(\frac{D}{2}\right)^2$, and the potential energy of both springs is $\frac{1}{2}k\left(\frac{D}{2}\right)^2$.

The initial potential energy of the monkey is MgH. We will call the height where the monkey stops h_k. The velocity of the monkey is zero, therefore the kinetic energy of the monkey is zero. The potential energy of the ape is Mgh_k. The energy of $Mg(H - h_k)$ is now added to the potential energy of the springs.

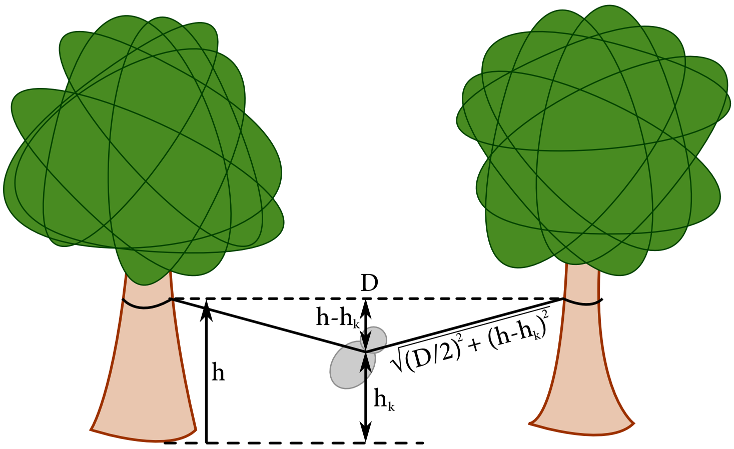


Figure 14: *Monkey in the lowest point*

Applying a bit of geometry, we get the length of the springs when the monkey stops $\sqrt{(h-h_k)^2 + \left(\frac{D}{2}\right)^2}$. Now we can state the consrvation of energy equations for this problem.

$$Mg\,(H-h_k) + 2\frac{1}{2}k\left(\frac{D}{2}\right)^2 = 2\frac{1}{2}k\left(\sqrt{(h-h_k)^2 + \left(\frac{D}{2}\right)^2}\right)^2,$$

$$Mg\,(H-h_k) + 2\frac{1}{2}k\left(\frac{D}{2}\right)^2 = 2\frac{1}{2}k\left((h-h_k)^2 + \left(\frac{D}{2}\right)^2\right),$$

$$Mg\,(H-h_k) = k(h-h_k)^2.$$

We now have a quadratic equation which roots are

$$h_k = h - \frac{Mg}{2k} \pm \frac{1}{2k}\sqrt{M^2g^2 + 4kMg(H-h)}.$$

We now need to determine, which solution makes sense. We can get this by solving special case for $H = h$:

$$Mg\,(h-h_k) = k(h-h_k)^2 \Rightarrow h_k = h - \frac{Mg}{k}.$$

This prooves that the one with minus sign is correct. Therefore, the solution is

$$h - \frac{Mg}{2k} - \sqrt{\left(\frac{Mg}{2k}\right)^2 + \frac{Mg}{k}(H-h)}$$

or

$$h - \frac{Mg}{2k}\left(1 + \sqrt{1 + \frac{4k}{Mg}(H-h)}\right).$$

26 In this problem we only need to calculate all moments of inertia and torques. Since all weights are point masses, the total moment of inertia can be calculated immediately:

$$I = mr^2 + 2mr^2 + \ldots + 12mr^2 = 78mr^2.$$

Then we need to calculate the torques:

$$N = \left(12 \cdot 0 + 11 \cdot \frac{1}{2} + 10 \cdot \frac{\sqrt{3}}{2} + 9 \cdot 1 + 8 \cdot \frac{\sqrt{3}}{2} + 7 \cdot \frac{1}{2} + 6 \cdot 0 - 5 \cdot \frac{1}{2} - 4 \cdot \frac{\sqrt{3}}{2} - 3 \cdot 1 - 2 \cdot \frac{\sqrt{3}}{2} - 1 \cdot \frac{1}{2}\right) mgr,$$

$$N = 6\left(2 + \sqrt{3}\right) mgr.$$

The result is thus $\varepsilon = \frac{N}{I} = \frac{2+\sqrt{3}}{13}\frac{g}{r}$.

27 Intuitively we expect the frequency of a source to change if its speed changes in our frame of reference. This is described by Doppler effect with the following formula used to calculate the frequency change:

$$f' = \frac{c \pm v_r}{c \mp v_s} f,$$

where c is the speed of sound, w_r is the velocity of the receiver and v_s is the velocity of the source. The signs depend on whether the source approaches or receeds from the receiver. In this case, we know that when the plane was approaching, the frequency was half an octave higher, this means

$$\sqrt{2} f = \frac{c}{c - v} f,$$

where v is velocity of the aeroplane. Rearranging this expression we find

$$v = \frac{2 - \sqrt{2}}{2} c.$$

If the aeroplane is moving away from the receiver, the frequency becomes

$$f' = \frac{c}{c + v} f;$$

substituting in for velocity we find

$$f' = \frac{4 + \sqrt{2}}{7} f.$$

28 We label the single distance as h and draw a picture. Let's look at a beam, coming from Adam's head and pointing towards Johny's eyes. The beam's way will break at the distance x from the pool's edge. With this, Adam's illusive height H can be simply found using triangle similarities:

$$H = h\frac{x}{h - x}.$$

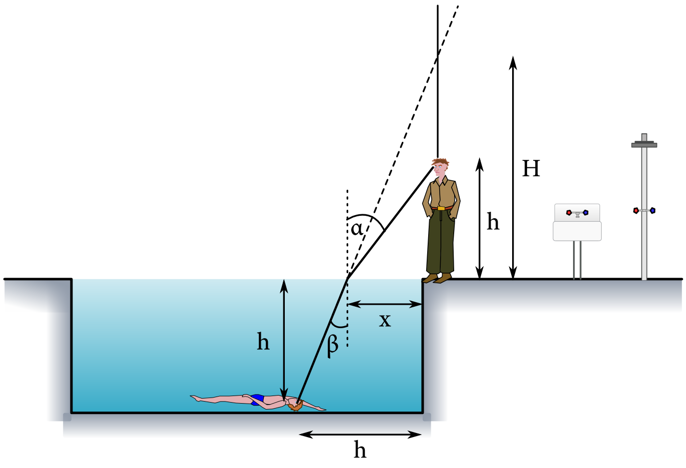


Figure 15: *Situation analysis*

In order to find x, we can use either Snell's law or Fermat principle

Snell's law

Writing Snell's equation

$$\sin\alpha = n\sin\beta,$$

$$\frac{x}{\sqrt{x^2+h^2}} = n\frac{(h-x)}{\sqrt{(h-x)^2+h^2}},$$

$$\frac{x^2}{x^2+h^2} = n^2\frac{(h-x)^2}{(h-x)^2+h^2},$$

$$\vdots$$

$$0 = \left(n^2-1\right)\left(2x^2h^2-2x^3h+x^4\right)+n^2\left(h^4-2xh^3\right).$$

This equation is sadly of 4th order, so it is extremely hard to solve. But there is still a way, if one is not afraid of numerics. Clearly, the simplest method one can use is the *binary search algorithm*. Firstly, we transform the above equation to the function: $f(x) = \left(n^2-1\right)\left(2x^2h^2-2x^3h+x^4\right)+n^2\left(h^4-2xh^3\right)$. Now we will watch how the sign of this function changes over some interval.

If the sign changes over some interval, there has to be the zero point somewhere between. [2] The strategy is based on the technique of shortening the interval until the zero point is found.

- In the very beginning, we just make a guess on that interval $[l;r]$. In our case, it's easy to guess since the solution must exist somwhere between 0 and $h = 2$ m.
- Now let's look at the result's sign at point $\alpha = \frac{l+r}{2}$. Depending on which sign we found, we will replace the middle point $\alpha = \frac{l+r}{2}$ with either left edge l or right edge r.
- We iterate these steps until the interval is so small, that the searched result won't change on first decimal place.

[2] This holds only for continuous functions – drawable with a single move.

After couple of itterations, we come to value $x = 1{,}18$ m, which give us the estimate for H interval from 2,8 m to 3,0 m (depending on rounding of metaresults)

Fermat principle

The Fermat principle claims that light is travelling along such path, which optimizes total time of traveling. In our case, we want to minimize the time of traveling beam along certain path:

$$t(x) = \frac{1}{c}\sqrt{x^2 + h^2} + \frac{n}{c}\sqrt{(h-x)^2 + x^2}.$$

Now the task is find such x, which would minimize the time. Since x lies between $x \in [0\text{ m}; 2\text{ m}]$, we can calculate the time 10 times for increasing x with step 0,2 m. This can be repeated within new-found interval and so on… With this, one can come to result $x = 1{,}18$ m, which give us for H values from 2,8 m to 3,0 m.

$\boxed{29}$ The smallest light flux, recognisable by Johny, can be derived using bulb info:

$$F_{\min} = \frac{0{,}04 \cdot 100\text{ W}}{4\pi\,(100\,000\text{ m})^2} = \frac{1}{\pi} \times 10^{-10}\text{ W}\,\text{m}^{-2}.$$

The solar constant $F_\odot$ expresses the solar energy flux in distance of $R_\oplus = 1\text{ AU} \doteq 1{,}5 \times 10^{11}\text{ m} \doteq 4{,}86 \times 10^{-6}$ pc. The power of visible light radiation is then $0{,}36 \cdot F_\odot 4\pi R_\oplus^2$ and the flux of visible light made by the Sun in distance D is $F = 0{,}36 \cdot F_\odot \frac{R_\oplus^2}{D^2}$. We are looking for such a distance, at which the flux will be equal to $F_{\min}$. Simple calculation reveals it as 19 pc.

$\boxed{30}$ It is known that charged particles follow circular trajectories in a magnetic field. From a balance of centripetal force and magnetic force, one can obtain a radius of the trajectory $r = \frac{mv}{QB}$, where m denotes mass of the particle and v its speed.

Diagonals of the square with edges a have to be tangents of the circular path of the particle, thus its radius is $\frac{\sqrt{2}a}{2}$.

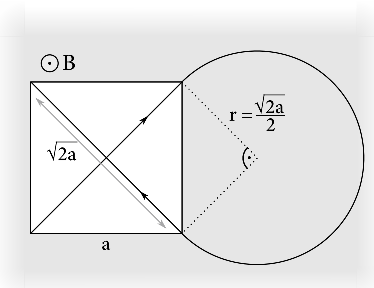


Figure 16: *Path of particle*

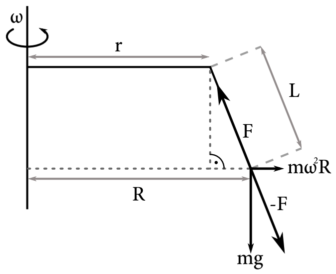
Time t links speed and lenght of the diagonal $\sqrt{2}a$: $v = \frac{\sqrt{2}a}{t}$. Substituting speed in the expresion for the radius, one obtains

$$m = \frac{QBr}{v} = \frac{QB\frac{\sqrt{2}a}{2}}{\frac{\sqrt{2}a}{t}} = \frac{QBt}{2}.$$

31 In the case of limiting angular velocity the flea has to act by maximum force of F to compensate for the forces in the rotating system, ie. gravitational and centrifugal force.

George has to be in equilibrium, but unlike the flea he is attached by the chain. This means the simmilarity of imaginary triangles in the picture.

$$\frac{R-r}{L} = \frac{m\omega^2 R}{F}.$$



Because the centrifugal and gravitational forces are perpendicular, we can state the

$$m\omega^2 R = \sqrt{F^2 - m^2 g^2}.$$

We can now remove all R from the first equation, thus we get

$$\frac{\frac{\sqrt{F^2-m^2g^2}}{m\omega^2} - r}{L} = \frac{\sqrt{F^2 - m^2 g^2}}{F}.$$

After a rather unpleasnt, but not conceptually difficult task of finding ω, we get the result

$$\omega = \sqrt{\frac{\sqrt{\frac{F^2}{m^2} - g^2}}{r + L\sqrt{1 - \frac{m^2 g^2}{F^2}}}}.$$

32 In order to maintain constant radiation levels, decayed atoms of potassium have to be refilled. We know that Rosalinda's activity is $A = 10^8$ Bq, which means that 10^8 atoms of potassium ^{40}K decay every second. These need to be refilled by consuming bananas. Each banana contains $N = \frac{mN_A}{M_m}$ atoms of radioactive potassium, where $M_m \approx 40\ \mathrm{gmol}^{-1}$ is the molar mass of ^{40}K.

When the monkey is in dynamic equilibrium, the number of decayed atoms must equal the acquired atoms, therefore $A \overset{!}{=} fN$ where f is the frequency at which Rosalinda eats bananas. The speed at which Rosalinda eats bananas is $v = Lf = \frac{LAM_m}{mN_A}$, where $L = 20$ cm. is the lenght of the banana.

Plugging in numbers, we find the speed at which Rosalinda has to eat bananas equals $v \doteq 2{,}2 \times 10^{-11}$ m/s = 22 pm/s.

33 It is known that a charged plate creates a homogenous electric field in its surroundings with intensity $\frac{\sigma}{2\varepsilon_0}$ and with a dirrection away from the plate. However, electric fields of two parallel plates charged by same charge compensate each other, so the overall electric force exerted on protons between plates is zero. Therefore, protons will reach all points of the space between the plates after certain time.

34 Firstly, lets revise the definitions of sound intensity and sound intensity levels. The formula for intensity is $I = \frac{P}{4\pi r^2}$, where P is the source power and r is the distance from the source. The sound intensity level is $L_I = 10 \log_{10} \frac{I}{I_0}$, where I_0 is the reference intensity. What does it mean?

When he listens to music on earphones, the right ear only hears the right earphone and the left ear only hears the left earphone. If the right earphone is boosted, it has no effect on what the left ear hears. In this case he needs a +3 dB correction on the right earphone. Hence the intensity of the right increases by a factor of $10^{\frac{3}{10}}$. Then the sound going into the right ear is suppressed by a factor of $k = 10^{-\frac{3}{10}} \doteq 0{,}5$.

The situation becomes more complicated if he uses loudspeakers because both ears hear the sound from both speakers. In this case Martin needs a +13 dB correction on the right speaker, which means the intensity ratio of the right and left speaker is $q = 10^{\frac{13}{10}}$. Denote the distance between the speakers by $2L$ and the distance between Martin's ears $2x = 15$ cm, then the intensities heard by each ear are as follows:

$$I_{\mathrm{L}} = \frac{P}{4\pi (L-x)^2} + \frac{qP}{4\pi (L+x)^2}, I_{\mathrm{R}} = k\left(\frac{P}{4\pi (L+x)^2} + \frac{qP}{4\pi (L-x)^2}\right).$$

However, in the problem it is claimed that both ears hear the sound with the same intensity. Hence we subtract these equations to get a quadratic equation L:

$$\frac{P - kqP}{4\pi (L-x)^2} = \frac{kP - qP}{4\pi (L+x)^2}, \sqrt{\frac{1-kq}{k-q}} = \frac{L+x}{L-x}, L = x\frac{1+\sqrt{\frac{1-kq}{k-q}}}{1-\sqrt{\frac{1-kq}{k-q}}}, L = x\frac{k-q+1-kq+2\sqrt{(1-kq)(k-q)}}{k-q-1+kq}.$$

After plugging in the values we find the distance between speakers to be $2L \doteq 79$ cm.

35 Let us denote the satellite's edge by a. As the problem statement suggests, the satellite can be assumed to be a perfect black body. This means all radiation incident on the surface is absorbed and the re-radiation follows the Stefan-Boltzmann law. It also implies that the total electromagnetic radiation power is proportional to the satellite's area and to fourth power of its temperature.

If thermodynamic equilibrium is to occur, the absorption and radiation powers must be equal. Radiation power is independent of the spatial orientation as the body is perfectly conductive and its total area is constant. Hence we can express the radiation power as $\sigma 6a^2 T^4$, where σ is the Stefan-Boltzmann constant and T is the satellite's surface temperature.

Because the radiation power is constant, equilibrium is determined solely by the amount of incoming radiation. The absorbtion power will be proportional to the effective surface area S. The equilibrium temperature can then be easily found by equating the radiation and absorption powers. Let F be the solar radiation flux[3], then

$$FS = \sigma 6a^2 T^4 \Rightarrow T = \sqrt[4]{\frac{FS}{\sigma 6a^2}}.$$

Minimum effective crossection area is achieved by positioning the cubic satellite such that one of its faces is rotated directly towards the sun, the crossection is then a^2.

[3] radiation power per unit area

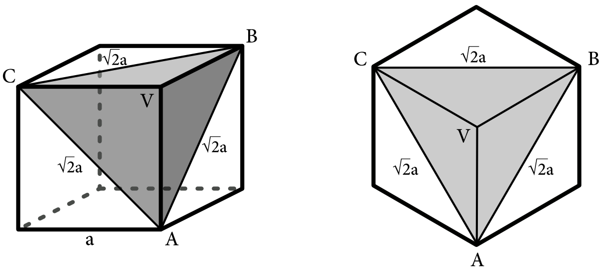


Figure 17: *A cube inclined to the Sun with the greatest possible surface*

Maximum possible effective crossection is obtained by aligning one of the body diagonals to point at the Sun. In other words, the plane defined by points A, B and C is perpendicular to the aforementioned line. Looking from the Sun, the satellite then appears as a regular hexagon with an inscribed equilateral triangle ABC of side $\sqrt{2}a$.

Area of a regular hexagon is equal to twice the area of the inscribed equilateral triangle (see picture). By basic geometry height of the triangle can be found to be $\frac{\sqrt{3}}{2}\sqrt{2}a$. Then the hexagon's area is equal to $\sqrt{3}a^2$.

Minimum and maximum obtainable temperatures are then

$$T_{\min} = \sqrt[4]{\frac{Fa^2}{\sigma 6a^2}},$$

$$T_{\max} = \sqrt[4]{\frac{F\sqrt{3}a^2}{\sigma 6a^2}}.$$

From these expressions follows the ratio

$$\frac{T_{\max}}{T_{\min}} = \sqrt[8]{3}.$$

A note about the cross section of a cube

A meticulous reader might have been curious whether there is a proof that the cross section of a cube is minimal when it is inclined with its face and maximal when its body diagonal points to the Sun.

A projection of a cube face – i.e. square – of unit size can be expressed as dot product of unit normal vector and unit vector determined by direction of projecting. From any direction can we see at most three faces of the cube, normal vectors of which are perpendicular to each other (denote them $\hat{x}$, $\hat{y}$ and $\hat{z}$). A total area of projection when looking from general direction (x, y, z) would be a sum of projection areas of individual faces

$$(|x|, |y|, |z|) \cdot (\hat{x} + \hat{y} + \hat{z}) = |x| + |y| + |z|.$$

A symetry of the cube implies that one needs to consider only one octant, e.g. the first one (so that $x, y, z > 0$). Therefore, the original problem can be transformed to the problem of function extreme finding. In our case, we need to find minimum and maximum of the function $\varphi(x, y, z) = x + y + z$ with respect to the aditional condition $x^2 + y^2 + z^2 = 1$, as a norm of the vector (x, y, z) is 1. The considered function reaches its

- minimum, if one of the vector coordinates is 1 and the remaining two coordinates equal to 0 what correspond to the projection axis perpendicular to a cube face
- maximum, if $x = y = z = \frac{\sqrt{3}}{3}$ what correspond to the projection in direction of a body diagonal.

36 The electron is in a potential well near helium nucleus. Another potential well is near the proton. We want to calculate the minimal energy to escape from the potential well near helium, ie. to break through the potential barrier between two wells.

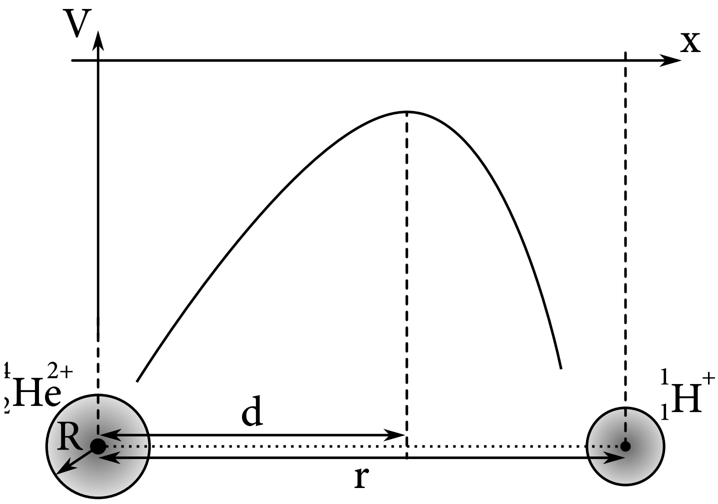


Figure 18: *The potential as a function of distance*

Firstly, we need to calculate the maximum of the potential barrier of d.When the electron reaches the potential barrier, the sum of all forces acting on it equals zero, thus the electrostatic force of the helium nucleus and the electrostatic force of the proton. Mathematicaly:

$$\frac{1}{4\pi\varepsilon_0}\frac{2e^2}{d^2} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{(r-d)^2}.$$

From this we get the maximum:

$$d = \left(2-\sqrt{2}\right) r.$$

Now we find a function of potential between the proton and the helium nucleus. Let x be the distance from the helium nucleus. The potential can be found by superposition:

$$V(x) = -\frac{1}{4\pi\varepsilon_0}\frac{2e}{x} - \frac{1}{4\pi\varepsilon_0}\frac{e}{r-x} = -\frac{e}{4\pi\varepsilon_0}\frac{2r-x}{x(r-x)}.$$

Finally, we can calculate electron needs in order to escape fro the reach of the helium nucleus. We can get this from difference of the potential energy of electron at the barier and the potential energy of electron at the helium core.

$$E = \frac{e^2}{4\pi\varepsilon_0}\left(\frac{2r-R}{R(r-R)} - \frac{\sqrt{2}}{\left(2-\sqrt{2}\right)\left(\sqrt{2}-1\right)r}\right).$$

Using $r = 11R$, we get:

$$E = \frac{e^2}{4\pi\varepsilon_0}\left(\frac{201-20\sqrt{2}}{110R}\right).$$

37 An infinite amount of capacitors may seem to be an infinite amount of work, but one needs to realize only a few things and it simplifies solving of this problem significantly. Firstly, Martin adds same components composed of 4 capacitors again and again, secondly, we can create Martin's infinite scheme from 2 infinite schemes joined to the elementary component. What a beauty of infinities! If you have not got it yet, contemplate a bit more and you may become enlightened.

Denote an overall capacitance of the scheme C_∞. We can redraw the scheme as follows:

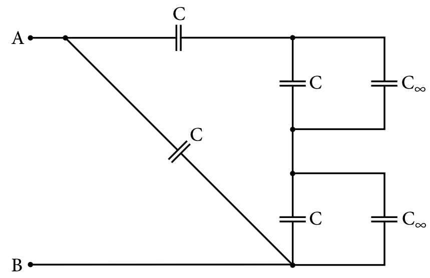


Figure 19: *Redrawn scheme*

It is only a series and parallel combination of capacitors. Capacitances of parallel capacitors are to be summed. In case of series capacitors, an inverted values of capacitances are to be summed. We obtain the following equation

$$C_\infty = \frac{C^2 + CC_\infty}{3C + C_\infty} + C.$$

It leads to a solving of a quadratic equation which is sattisfied by

$$C_\infty = \frac{-C \pm \sqrt{C^2 + 16C^2}}{2}.$$

As the capacitance cannot be negative, the correct solution is

$$C_\infty = \frac{\sqrt{17} - 1}{2} C.$$

38 Since we are interested in the small oscillation period, it is necessary to describe motion of all frame parts via their centers of mass. The short rods of length a, hanging on wall, will perform rotational motion about their fixed points. In other words, their centers of mass will move on circles of radius $\frac{a}{2}$. In the case of longer rod of length b, the center of mass will rotate on circle of radius a, as depicted on picture bellow

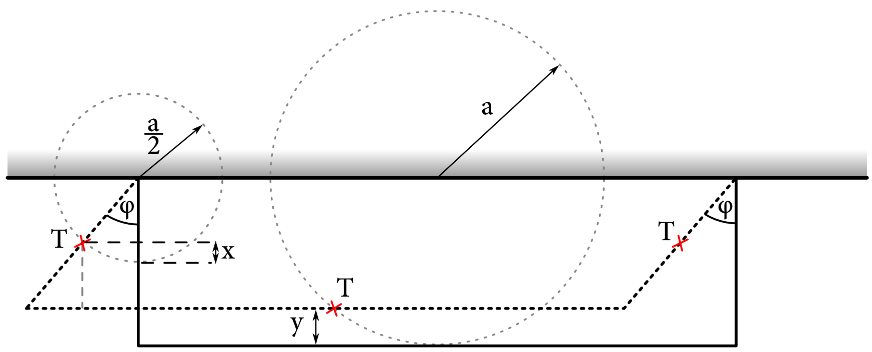


Figure 20: *Angle φ is quite big, just for illustration purpose.*

One way to solve this problem is to calculate all forces and torques, but that will be time consuming and boring. A more interesting way is to do it using energies. Let's start with the kinetic one. In general, the moment of inertia of a rod of length l with respect to its ending point is $\frac{1}{3}ml^2$, so its kinetic energy would be:

$$E^a_{kin} = \frac{1}{2}I_a\omega^2 = \frac{1}{6}\lambda a^3\omega^2,$$

where ω is an angular velocity. It would be too complicated to calculate the whole motion of longer rod. However, if one realizes that all parts of longer rod have to move with the same velocity, which is equal to the angular velocity of short rods ends, the kinetic energy of longer rod can be written as

$$E^b_{kin} = \frac{1}{2}m_b v^2 = \frac{1}{2}\lambda b a^2\omega^2,$$

so the total kinetic energy of the frame is

$$E_{kin} = \left(\frac{1}{3}a + \frac{1}{2}b\right)\lambda a^2\omega^2.$$

Now let's calculate the potential energy change, whene deviated by an angle φ. The centers of mass of the shorter rods will raise by $x = \frac{a}{2}(1-\cos\varphi)$ and by $y = a\,(1-\cos\varphi)$ for the longer one. Total potential energy change is then

$$E_{pot} = 2\lambda a\, gx + \lambda bgy = \lambda a(a+b)g(1-\cos\varphi).$$

When dealing with small oscillations, potential energy should depend quadratically on deflection - then we obtain simple motion equations. Using Taylor expansion around minimum, we can simply replace the term $1-\cos\varphi$ with $\frac{\varphi^2}{2}$. Then we obtain

$$E_{pot} \approx \lambda a(a+b)g\frac{\varphi^2}{2}.$$

Now we are able to build motion equation, solve it and achieve the wanted period. Although, we can do it much easier, since it is an analog of a classic oscillator. Note that the energies for a classic oscillator are $E_{\text{kin}} = \frac{1}{2}mv^2$, $E_{\text{pot}} = \frac{1}{2}kx^2$ and the period of small oscillations can be calculated as $T = 2\pi\sqrt{\frac{m}{k}}$. Analogously, we can now assign values from our expressions to m and k and simply calculate the period. The final result is:

$$T = 2\pi\sqrt{\frac{\left(\frac{1}{3}a+\frac{1}{2}b\right)\lambda a^2}{\frac{1}{2}\lambda a\,(a+b)\,g}} = 2\pi\sqrt{\frac{\left(\frac{2}{3}a+b\right)a}{(a+b)\,g}}.$$

$\boxed{39}$ Consider a planet of mass M and its moon of mass $m = \frac{M}{10}$ in the mutual distance R. Masses of these two objects are similar, therefore, one must consider they orbit around their mutual centre of mass.

Firstly, let's find distances of these objects from the centre of mass. Denote the distance of the moon from the centre of mass r. Then

$$r = \frac{M}{M+m}R, \quad R-r = \frac{m}{M+m}R.$$

Secondly, let's find velocities of the planet and the moon. From the balance of a centripetal force and gravity acting on the moon, one can obtain

$$G\frac{Mm}{R^2} = m\frac{v_0^2}{r} \qquad \longrightarrow \qquad v_0 = \sqrt{\frac{G}{R\,(M+m)}}M.$$

The same condition aplied for the planet gives

$$G\frac{Mm}{R^2} = M\frac{V_0^2}{R-r} \qquad \longrightarrow \qquad V_0 = \sqrt{\frac{G}{R\left(M+m\right)}}m.$$

Suddenly, Gandalf appeared and changed the mass of the planet, while its velocity remained untouched. Thus, there is a planet of mass M' moving with velocity V_0 and the moon with mass m moving in the opposite dirrection with velocity v_0. Energy of this system is

$$E = \frac{1}{2}M'V_0^2 + \frac{1}{2}mv_0^2 - G\frac{M'm}{R} = \frac{1}{2}M'\frac{Gm^2}{R\left(M+m\right)} + \frac{1}{2}m\frac{GM^2}{R\left(M+m\right)} - G\frac{M'm}{R}.$$

Consider the entire system as a whole. As there are no external forces, the momentum of the system must be preserved. That implies the velocity of the centre of mass of the system remains constant. Let's find this velocity

$$v_T = \frac{mv_0 - M'V_0}{M'+m} = \sqrt{\frac{G}{R\left(M+m\right)}}\frac{m\left(M-M'\right)}{M'+m}.$$

What is a condition for escape of the moon from the planet's gravitationl field? Usually, if a stationary planet is considered, zero kinetic energy of the moon at the infinite distance is required. However, we consider the moving planet and if the moon stopped in the infinity, the potential energy of the system would keep changing due to the planet movement, therefore, the stop of the moon would not correspond to the minimum of energy and thus the moon would start accelerating again. The condition for the moon escape must be zero mutual velocity of the moon and the planet at the infinite distance. Potential energy at the infinite distance is zero, so the system will have only kinetic energy. However, we know that the velocity of the centre of mass cannot change, therefore the energy of the system at infinity is

$$E = \frac{1}{2}\left(M'+m\right)v_T^2 = \frac{Gm^2\left(M-M'\right)^2}{2R\left(M+m\right)\left(M'+m\right)}.$$

As the energy must be preserved, we obtain

$$\frac{1}{2}M'\frac{Gm^2}{R\left(M+m\right)} + \frac{1}{2}m\frac{GM^2}{R\left(M+m\right)} - G\frac{M'm}{R} = \frac{Gm^2\left(M-M'\right)^2}{2R\left(M+m\right)\left(M'+m\right)}.$$

It leads to the result

$$M' = \frac{M-m}{2}.$$

As $m = \frac{M}{10}$, we finally obtain

$$M' = \frac{9}{20}M.$$

40 Firstly, let's think about the force acting on the mast, trying to rotate it. There is tension T acting in place, where rope is attached to the mast. In every moment, this force has to compensate Johny's weight $mg\cos\varphi$ and centrifugal force mu^2/L, since Johny is rotating on circle of radius L. In this u is Johny's actual velocity. Therefore:

$$T = mg\cos\varphi + \frac{mu^2}{L}.$$

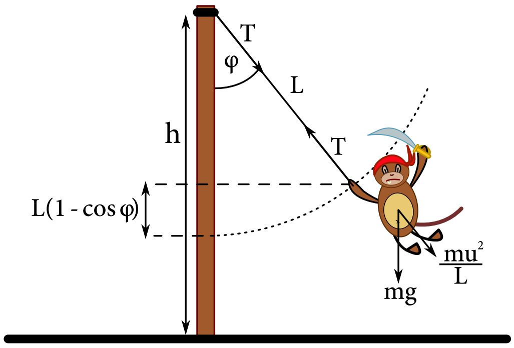


Figure 21: *Situation with sketched forces*

Velocity u can be determined from conservation law of energy. The zero reference level of potential energy is set to the place where free rope would reach. If Johny is deviated by an angle φ, potential energy would raise by $mgL\,(1-\cos\varphi)$. In the very beginning, Johny had zero potential energy and non-zero kinetic energy $mv^2/2$. Conservation law yields:

$$\frac{mv^2}{2} = \frac{mu^2}{2} + mgL\,(1-\cos\varphi)\,,$$

$$mu^2 = mv^2 - 2mgL\,(1-\cos\varphi)\,,$$

Torque acting on mast, as a function of angle φ, is then

$$M(\varphi) = Th\sin\varphi = h\left(mg\cos\varphi + \frac{mv^2 - 2mgL\,(1-\cos\varphi)}{L}\right)\sin\varphi,$$

$$M(\varphi) = \frac{h}{L}\left(mgL\cos\varphi + mv^2 - 2mgL\,(1-\cos\varphi)\right)\sin\varphi,$$

$$M(\varphi) = \frac{h}{L}\left(mv^2 - 2mgL + 3mgL\cos\varphi\right)\sin\varphi.$$

Maximum of this function can be found by using differentiation technique. We obtain

$$\frac{\mathrm{d}M(\varphi)}{\mathrm{d}\varphi} = \frac{h}{L}\left(\left(mv^2 - 2mgL\right)\cos\varphi + 3mgL\left(-\sin^2\varphi + \cos^2\varphi\right)\right),$$

Using trigonometrical identity $\sin^2\varphi + \cos^2\varphi = 1$, one can rewrite the differentiation to the form containing orders of $\cos\varphi$:

$$\frac{\mathrm{d}M(\varphi)}{\mathrm{d}\varphi} = \frac{h}{L}\left(\left(mv^2 - 2mgL\right)\cos\varphi + 3mgL\left(-1 + 2\cos^2\varphi\right)\right),$$

If maximum is present, the differentiation should equal zero and therefore, we have to deal with quadratic equation for $\cos\varphi$

$$\frac{h}{L}\left(\left(mv^2 - 2mgL\right)\cos\varphi - 3mgL + 6mgL\cos^2\varphi\right) = 0,$$

$$\left(v^2 - 2gL\right)\cos\varphi - 3gL + 6gL\cos^2\varphi = 0.$$

By solving we obtain two possible results:

$$\cos\varphi = -\frac{v^2-2gL}{12gL}\left(1 \pm \sqrt{1+\frac{1}{2}\left(\frac{12gL}{v^2-2gL}\right)^2}\right).$$

Since we know that Johny wasn't moving with enough velocity to jump over the top of mast, it holds $v^2 < 2gL$. We can now formulate the result as:

$$\cos\varphi = \frac{2gL-v^2}{12gL}\left(1 \pm \sqrt{1+\frac{1}{2}\left(\frac{12gL}{v^2-2gL}\right)^2}\right).$$

Apparently, we obtained two results. We can pick the correct (physical) one by using the fact that, the second differentiation should be negative if the first differentiation equals zero. After doing this, we are picking the result with minus sign. Therefore, the bottom part of mast is most stressed, when the angle equals

$$\varphi = \arccos\left(\frac{2gL-v^2}{12gL}\left(1 + \sqrt{1+\frac{1}{2}\left(\frac{12gL}{v^2-2gL}\right)^2}\right)\right).$$

The result can be also written in this form:

$$\varphi = \arccos\left(\frac{2gL-v^2}{12gL} + \sqrt{\left(\frac{2gL-v^2}{12gL}\right)^2 + \frac{1}{2}\,\frac{2gL-v^2}{12gL}}\right).$$

Answers

1 George's brother is faster and beats Michael's brother by 10 690 nanopuerperia. Accept results in the range of [10 401; 10 827] nanopuerperia.

2 32 %

3 36 m

4 $\frac{\rho_i a^3}{\rho_w \pi R^2}$

5 $a + \frac{F}{m}$

6 7 Ω, 12 Ω a 15 Ω. *Accept only if all three resistances are correct.*

7 5570 km

8 $10{,}24\ g = \frac{256}{25}\ g$

9 45°

10 $20\sqrt{3}\,\mathrm{ms^{-1}} \doteq 35\,\mathrm{ms^{-1}}$

11 $\frac{a}{2f}$

12 1764 m

13 0,2 kg

14 $\frac{5\pi}{3\pi+4}$ cm $\doteq$ 1,17 cm

15 $\frac{k_A}{k_A+k_B}n_0$

16 $d\sin\left(\arctan\left(\frac{w}{v}\right)\right) = d\frac{w}{\sqrt{v^2+w^2}}$

17 14 m

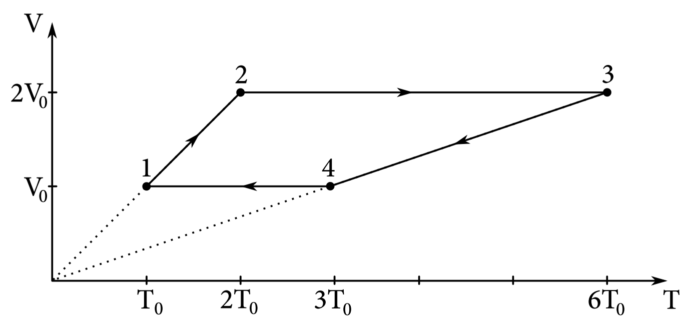
18 Resistance of the scheme can reach values in the range of $\left[\frac{3}{5}R; \frac{5}{3}R\right]$.

19 2158 cm³, *also accept* 2284 cm³ *which can be obtained if* $g = 9{,}81\,\mathrm{ms^{-2}}$ *is used.*

20 $\frac{(m+M)g}{4}$

21 45 Ω

22 *Heed that the submited graphs are labeled with all important values on axes, arrows determinating the course of the process have correct orientation and isobars lays on straight lines containing the origin.*



23 $\sqrt[3]{\frac{M}{N_A\rho}\frac{F}{SE}}$

24 $\frac{1}{2}$

25 $h - \frac{Mg}{2k}\left(1 + \sqrt{1 + \frac{4k}{Mg}(H-h)}\right) = h - \frac{Mg}{2k} - \sqrt{\left(\frac{Mg}{2k}\right)^2 + \frac{Mg}{k}(H-h)}$

26 $\frac{2+\sqrt{3}}{13}\frac{g}{r} \doteq 0{,}287\frac{g}{r}$

27 $\frac{4+\sqrt{2}}{7} f = \frac{2}{4-\sqrt{2}} f \doteq 0{,}77\, f$

28 *Approve results in the range of* 2,8 – 3,0 m.

29 19 pc

30 $\frac{QBt}{2}$

31 $\sqrt{\frac{F}{m}\frac{\sqrt{\frac{F^2}{m^2}-g^2}}{rF+L\sqrt{F^2-m^2g^2}}} = \sqrt{\frac{\sqrt{\frac{F^2}{m^2}-g^2}}{r+L\sqrt{1-\frac{m^2g^2}{F^2}}}} = \sqrt{\frac{1}{\frac{r}{\sqrt{\frac{F^2}{m^2}-g^2}}+\frac{mL}{F}}}$

32 22 pm/s = $2{,}2 \times 10^{-11}$ m/s

33 The entire area between the plates.

34 79 cm

35 $\sqrt[8]{3}$

36 $\frac{e^2}{4\pi\varepsilon_0}\left(\frac{201-20\sqrt{2}}{110R}\right) \doteq 0{,}125\frac{e^2}{\varepsilon_0 R} \doteq \frac{3{,}62\times 10^{-28}\ \mathrm{J}}{R} \doteq \frac{2{,}26\ \mathrm{neV}}{R}$

37 $\left(\sqrt{\frac{17}{4}}-\frac{1}{2}\right)C \doteq 1{,}56\, C$

38 $2\pi\sqrt{\frac{\left(\frac{1}{3}a+\frac{1}{2}b\right)\lambda a^2}{\frac{1}{2}\lambda a(a+b)g}} = 2\pi\sqrt{\frac{\left(\frac{2}{3}a+b\right)a}{(a+b)g}}$

39 $\frac{9}{20}M$

40 $\arccos\left(\frac{2gL-v^2}{12gL}\left(1+\sqrt{1+\frac{1}{2}\left(\frac{12gL}{v^2-2gL}\right)^2}\right)\right) = \arccos\left(\frac{2gL-v^2}{12gL}+\sqrt{\left(\frac{v^2-2gL}{12gL}\right)^2+\frac{1}{2}}\right)$