Milí čitatelia,

v rukách držíte zbierku úloh 21. ročníka Fyzikálneho Náboja. V zbierke sa nachádzajú všetky úlohy, s ktorými ste sa v roku 2018 mohli na súťaži stretnúť. K úlohám prikladáme aj vzorové riešenia, z ktorých sa môžete mnohé naučiť. Ak by ste danému vzorovému riešeniu nerozumeli, neváhajte sa nám ozvať, všetko objasníme.

Fyzikálny Náboj po minulom roku pokračuje vo svojej medzinárodnej tradícii. V roku 2018 sa do Náboja zapojili okrem Bratislavy takisto mestá Košice, Praha, Ostrava, Budapešť a Gdańsk. Výsledky vzájomného súboja si môžete pozrieť na našich stránkach.

Táto zbierka by nikdy nevznikla bez výraznej pomoci mnohých ľudí, ktorí sa koniec koncov podieľali na celom vývoji Fyzikálneho Náboja. Všetci sme študentmi Fakulty matematiky, fyziky a informatiky Univerzity Komenského a väčšina z nás sa aj aktívne podieľa na organizovaní Fyzikálneho korešpondenčného seminára (FKS).

FKS je korešpondenčný typ fyzikálnej súťaže. Zhruba raz za mesiac zverejňujeme rôzne zaujímavé fyzikálne úlohy, ktorých riešenia nám posielate do určených termínov. My vám za to dávame adekvátne body a tých najlepších pozývame koncom zimného a letného polroka na týždňové zážitkové sústredenie. Viac informácií nájdete na stránke `https://fks.sk/`.

Za finančnú pomoc ďakujeme firmám ESET a PosAm a za medzinárodnú spoluprácu lokálnym organizátorom: Róbert Hajduk (za Košice), Daniel Dupkala (za Prahu), Lenka Plachtová (za Ostravu), Ágnes Kis-Tóth (za Budapešť) a Kamil Żmudziński (za Gdańsk). V mene celého organizačného tímu veríme, že ste si v roku 2018 Fyzikálny Náboj užili a dúfame, že vás všetkých uvidíme aj o rok! Či už v roli súťažiacich, alebo organizátorov (v prípade, že už budete vysokoškolákmi).

Jaroslav Valovčan

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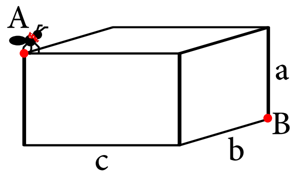
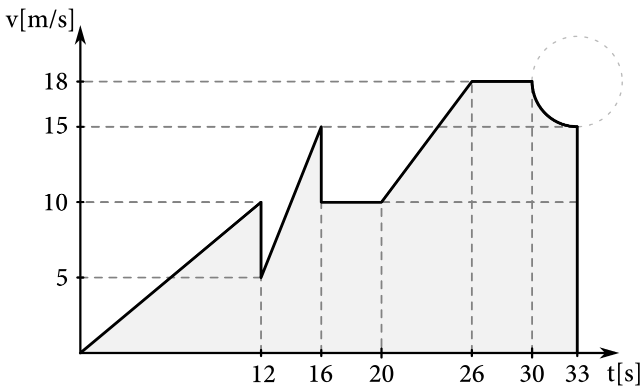
Výsledky súťaže, archív úloh a ďalšie informácie nájdete na stránke `https://physics.naboj.org/`.

Problems

1 Chris the ant sits on a vertex of a rectangular box with side lengths a, b and c, where $a < b < c$. The ant wants to crawl from vertex A to vertex B as quickly as possible. Chris' crawling speed is v. How long will his trip take?



2 Andrew learns to ride a unicycle. The plot shows his instantaneous speed against time during his ride. Determine Andrew's average speed.



3 Michael and Martin went hitchhiking to Krakow. Michael stands near a road where cars are passing with frequency 50 cars per minute in the correct direction and Martin stands at another place with frequency 25 cars per minute.

The journey to Krakow takes 100 minutes from Michael's spot and 90 minutes from Martin's one. Statistically, who arrives in Krakow sooner and by how much? Based on experience, they know that on average every 200$^{\text{th}}$ car picks up hitchhikers.

4 Grandma Justine discovered a huge bowl of unknown alcoholic drink in her cellar. She found out that it contains 40 percent of alcohol by volume and the rest is water. What percentage of alcohol by mass does it contain if density of pure ethanol is 790 kg/m^3?

Round the result to tenths of percent by mass.

5 John travels from Glasgow to Edinburgh. He accelerates at a constant rate from rest to the speed limit, keeps the speed and at the end of the journey slows down constantly until he stops. Acceleration and deceleration together took time t_1. On his way back, he is in a hurry, but he cannot go faster than the maximum speed limit, so the only thing he can do is to accelerate and decelerate more rapidly. Acceleration and deceleration on his way back took time $t_2 < t_1$. What was the difference between his travel times?

6 A garden sprinkler sprinkles water with density ρ from a nozzle with radius r upwards with velocity v. What is the mass of the water in the air at any moment?

7 George is squirming. He dares not hope. However, he gathers his courage and approaches the sales desk.

"Have you got hollow bricks?" he asks.

Surprisingly, the salesman does not call security as usually and answers with a smile on his face:

"Of course! What size would you like?"

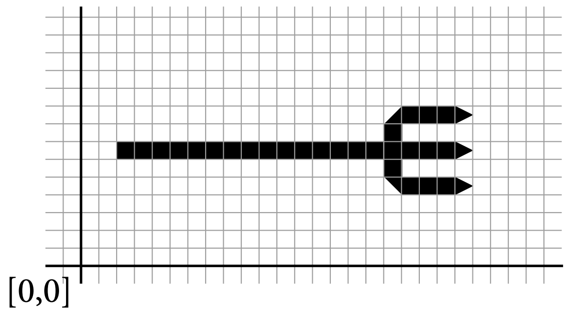
George can chose from perfectly hollow (completely empty) bricks with edge lengths of k, $2k$ and $4k$. Surface density of the bricks' material is 0.04 kg/m^2. However, George wants his first hollow brick to be special. He wants it to float freely in the air, whose density is 1.3 kg/m^3. What must the value of the parameter k be so that his wish is fulfilled?

The result should be rounded to centimeters.

8 On a hot summer day, Arthur poured his favourite sparking drink into a glass. A layer of foam formed at its top. The top of the foam is descending with speed u and the drink level is rising with speed v. What is the volume fraction of air in the foam?

9 Physics Náboj could expand to Asia, Australia... even the oceans! But first of all, we need a suitable sceptre. Find the coordinates of the centre of mass of the trident. The length of the side of a square in the grid is a.

Submit the result as a pair of fractions.



10 In the high tower of Minas Tirith, a minor elevator accident happened. The elevator cables broke and the cabin fell down from a height of 3 m. A poor hobbit was in the cabin at the moment. Fortunately, he was a skilled applied physicist. He knew the force of the impact would be mitigated a bit if he jumped up just before the crash. When standing on ground, he is able to jump to the height of 0.7 m.

From what height would he have to fall outside the elevator, in order to achieve the same impact velocity as inside the lift after the jump? The hobbit jumps at the last possible moment and his mass is negligible compared to the mass of the elevator cabin.

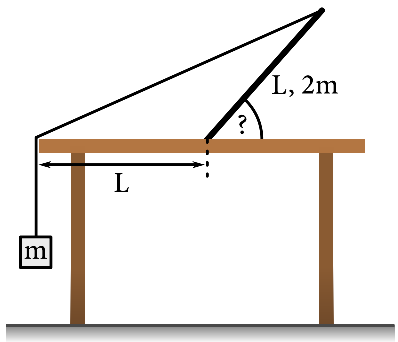
The result should be rounded to cm.

11 Kate was feeling lonely. She decided to put her sculptural talent to use and modelled a twin of hers. Despite the indisputable effort, the result was indistinguishable from a cylinder with radius 0.3 m, height 1.8 m and density 200 kg/m^3. However, it did not prevent her from taking the twin to the public. She took her to the swimming pool.

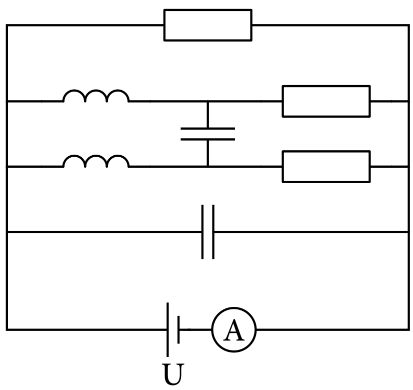
In the pool, she submerged the cylinder into the water, so that its upper base was at the water level. Then she released it and the buoyancy force pushed it upwards. What was the maximum height of the upper base above the water level during the jump?

Assume the cylinder remains in an upright position during the jump. Neglect the effects of water flow and surface tension.

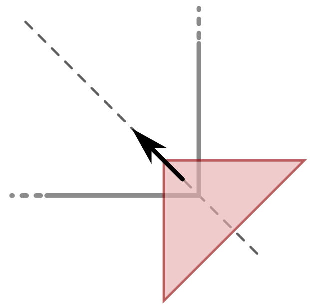
12 A rod of length L and mass $2m$ is attached to a table at one end so that it can rotate freely about the horizontal axis. At the other end of the rod, a massless rope is bound and hangs freely over the table's edge. At its hanging end, a counterweight of mass m is attached. The distance between the edge of the table and the joint connecting the rod to the table is L as well. What is the angle between the rod and the table if the system remains in an unstable equilibrium?



13 An electric circuit consists of resistors with resistivity R, capacitors with capacity C, coils with inductance L, ideal ammeter and ideal conductors. The circuit was connected to a source of voltage U. What is the current shown by the ammeter?



14 Matthew has decided to cover his summerhouse with a roof. He wants to put a right-angled isosceles triangle on the top of two perpendicular walls. He needs to place the roof as far along the diagonal as possible, as shown below. What fraction of the roof is now covering the interior of the summerhouse?



15 Adventurer Joseph got lost in a desert. The only piece of equipment he carried with him was a cuboid made of glass with square base of length 10 cm, height 20 cm and refractive index 1.5. For reasons unknown to Joseph, its upper base was painted black.

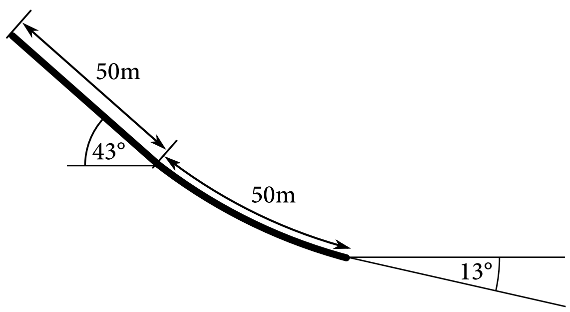
After a few days of endless wandering, Joseph noticed that the shadow cast by the painted base was much smaller than the actual base. What is the maximum altitude of the Sun in the sky so that the base does not cast any shadow?

16 Dory the fish is slowly swimming in the sea at a depth of 1 km, and as usual, she is meditating about the meaning of life. The temperature at that depth is 4 °C. Dory releases a bubble with radius 2 cm, which promptly starts its ascent to the surface. What will the radius of the bubble be just below the surface, where the water temperature is 18 °C? Assume that seawater is incompressible and the contribution of capillary pressure to the total pressure in the bubble is negligible. Furthemore, the air inside the bubble is immediately heated up to the temperature of surrounding water.

Submit the result rounded to mm.

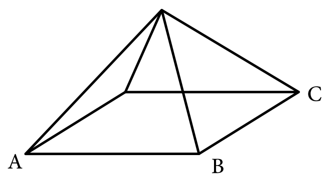
17 Philip decided to try out adrenaline sports. He chose ski jumping last winter. He practised on the ski jumping hill that is shown in the picture below. The in-run starts with a 50 m long inclined plane with a slope of 43° which transits continuously into a circular part with an arc length of 50 m. The take-off table forms an angle of 13° with the horizontal direction. What was the maximum g-force Philip experienced during his motion down the jumping hill?

G-force is defined as the acceleration of the body with respect to free fall.



18 Adam likes bungee jumping. When his stomach is empty, his mass is M and the period of his oscillations is T. How many pancakes must he eat in order to extend the period of his movement by one half, if one pancake has mass m?

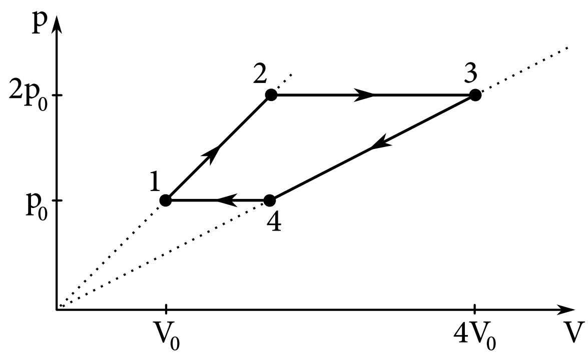
19 Catherine spent her holiday in Egypt this year. After she returned, she soldered a pyramid composed of eight identical wires with linear resistivity ξ and length L, as shown in the picture below. What is the ratio of resistances between the pairs of vertices A–B and A–C?



20 Martin is an astronomer. His quest to survey the skies led him to various exotic countries. As he lay directly on the equator, he noticed a bright artificial satellite approaching him from the west. It took exactly 8 minutes for the satellite to fly from the western to the eastern horizon. How high above the surface was its orbit if we assume its trajectory was circular? Do not forget to include the effects of the Earth's rotation.

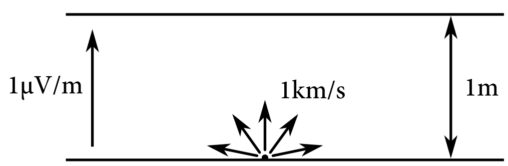
This problem does not have an analytical solution. Results that differ from an exact solution by no more than 1 km *will be accepted.*

21 An ideal gas with initial pressure p_0, volume V_0 and temperature T_0 underwent a cyclic process depicted in this pV diagram. Redraw this cycle to a VT diagram. The amount of the gas did not change during the process. Do not forget to mark all important values on the axes.

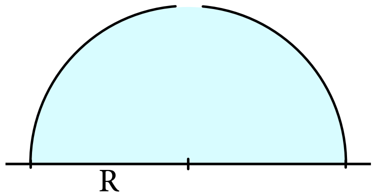


22 An isotropic source of electrons is placed on the surface of the bottom electrode of an infinite planar capacitor. The distance between the electrodes is 1 m and the electric field intensity is 1 µV/m. Electrons fly out with speed 1 km/s in all directions. What is the total area of the electrodes that can be hit by electrons?

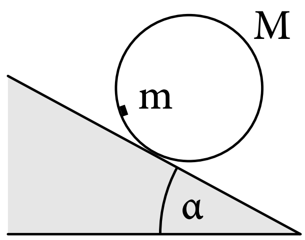
The result should be rounded to m^2.



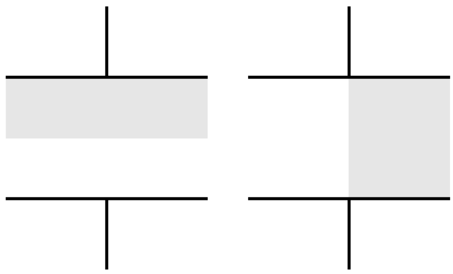
23 Caroline will not cook anymore! She put a hemispherical hollow bowl with radius R upside down on the table and drilled a hole into its middle. Then she started to pour water with density ρ throughout the hole. Before the bowl could overflow, water lifted it up and leaked out under the edge. What was the maximal possible mass of the bowl?



24 A piece of cheese of mass m is placed inside an empty cylindrical barrel of mass M. The coefficient of friction between the barrel and the cheese is μ. It was placed on a steep hill. What is the maximum angle of the slope α if the barrel remains on the slope at rest? The friction between the barrel and the hill is strong enough for the barrel not to slip down the hill.



25 We inserted a dielectric material with relative permittivity ε between the plates of two identical capacitors. We filled the top half of the first capacitor and the right half of the second one. What is the ratio of capacitances of the first and second capacitor?

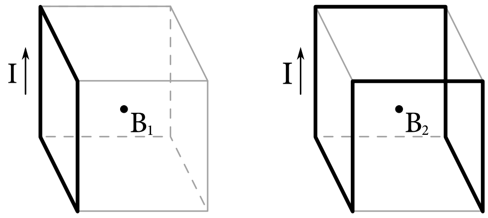


26 Martin, Nina and Oliver climbed the peak of Tarnica. In an attempt to silence Oliver for at least a while, Martin proposed a game based on imitating a simple anemometer. The trio positioned themselves on three of four vertices of a square 90 m in side, so that Martin and Nina were on opposite vertices. First, Martin and Nina shout and Oliver listens and measures the time required by the sound to reach him. The propagation time of Martin's shout was 255 ms and 285 ms$ of Nina's.

Unfortunately, Martin's plan had one shortcoming. In order to ensure the operation of the anemometer, they had to listen to Oliver afterward. The propagation time from Oliver to Martin was 304 ms and to Nina 272 ms. What was the wind velocity on the peak of Tarnica? Assume that the speed of sound is unknown.

Submit your result accurate to tenths of m/s.

27 Resistor networks in the shape of cubes are no fun anymore. Instead of building one of those, Duke decided to measure the magnitude of magnetic fields. When current flowed around a single face of a cube, he measured a magnetic field of intensity B_1. Then Duke bent the wire into a new shape. How many times is the magnetic field in the second circuit stronger compared to the first one?

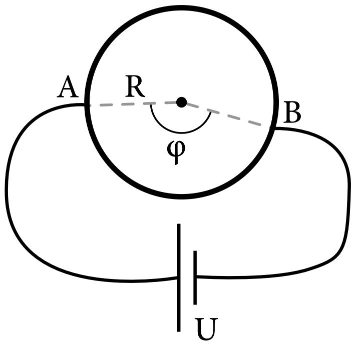


28 Jimmy bought a carton of milk with square base of length 1 dm and height 3 dm. He wants to drink some of the milk and then place it on an inclined plane with inclination 30°. What is the maximum volume of the milk he can leave in the carton so that it does not tip over?

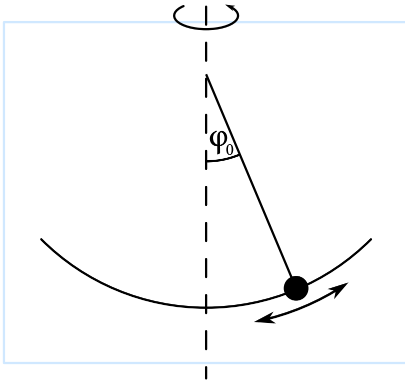
29 A planet is slowly orbiting its star on a circular trajectory. However, the calm space neighborhood is no longer what it used to be: pesky asteroids keep flying around, the star shines way too brightly and the nearby civilization has begun launching more and more singing electric cars. No wonder the planet loses its temper and violently splits into two pieces.

Both pieces keep moving in the original orbital plane, but their trajectories are now parabolic with perihelia at the position of the planet at the moment of explosion. What is the ratio of their masses (lighter to heavier)?

30 Sméagol was playing with old electronic components on the banks of his underground lake. He took his favourite golden ring with radius R and linear resistivity λ and connected it to voltage U. The points of connection of wires to the ring formed an angle φ with respect to the centre of the ring. He measured the magnetic induction at its center. How much is it?



31 Mike was having fun with a mathematical pendulum. He placed it between two parallel plates to restrict its motion to a plane. Then he sent it spinning together with the plates about the vertical axis with a constant angular velocity. The pendulum acquired an equilibrium position deflected by an angle $\varphi_0 \gg 0$ from the vertical direction. Then he tapped the bob slightly, and it started to oscillate about the equilibrium. Find the ratio of the pendulum's period about the vertical axis to the period of these small oscillations.



32 Crêpearth is a mythical world in the shape of an infinite flat disc. Above it, a spherical Sun floats at an altitude of $h = 500$ km. Its radius is $R_\odot = 10$ km and its temperature is $T_\odot = 5777$ K. Somewhere on Crêpearth a habitable zone has formed, where thermal equilibrium is reached within the temperature range of 0 – 30 °C. What is its area?

Both Crêpearth and the Sun are perfectly black bodies. Crêpearth does not conduct heat and only radiates upwards. You may neglect the angular size of the Sun as viewed from the surface. Only solutions that differ from the exact value by no more than $10\,000\ \mathrm{km}^2$ *will be accepted.*

33 An astronaut is standing on a small asteroid with mass M and radius R. After several hours, he realizes the surface of the asteroid is not exactly an exciting place. There is no wi-fi and as he is wearing a spacesuit, biting nails is also out of question.

To keep his morale high, he skips around a bit. He jumps at an angle of 45° with respect to the surface and after a short while, he lands at a distance equal to one fourth of the circumference of the asteroid. How much time did his flight take?

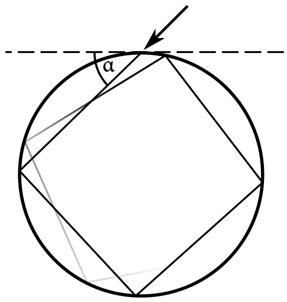
The asteroid is much heavier than the astronaut.

34 The unfortunate driver Martin ran a red light and was stopped by a policeman. He tried to fool him by saying that the red light appeared green to him. Now you need to help the policeman. How fast would the driver have needed to drive to have seen a red light with frequency 660 nm as a green light with frequency 550 nm?

Submit your answer as a fraction of the speed of light.

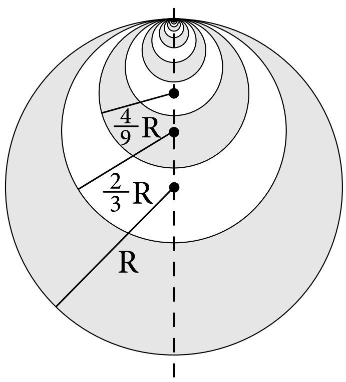
35 A tree-frog Katy and a toad Mike are hanging over a pulley. Both of them are at distance H from the pulley. Mike is twice as heavy as Katy. In order to keep the whole system balanced, a counterweight has been attached to Katy's end of the rope. When Mike, despite multiple warnings, does not stop croaking, Katy starts climbing the rope with speed v with respect to the rope. Which of the frogs will reach the pulley first and how far will the other frog be at that moment?

36 Rapper Saymon had earned way too much money and bought a luxurious cabriolet with perfectly reflecting chrome-steel wheels of radius R. When he shines a laser into the wheel in the plane perpendicular to its axis, the beam keeps bouncing around the inner surface for an infinitely long time. What is the **tensile** force acting on the wheel if the power of Saymon's laser is P, the laser was shining inside for time t and the angle between the beam and the tangent to the wheel's surface was α?

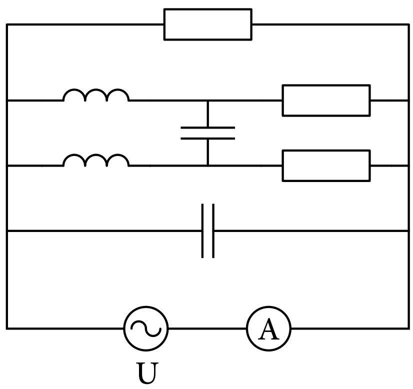


37 Samuel bought an enormous onion. Sadly, he forgot it out on his balcony and a part of it dried up. Now, the onion has a spherical shape with radius R and a spherical hole with radius $\frac{2}{3}R$ inside. In this hole, there is a full sphere with radius $\left(\frac{2}{3}\right)^2 R$, which has a spherical hole with radius $\left(\frac{2}{3}R\right)^3$ in it. There is another full sphere inside this hole, and so on and so on…

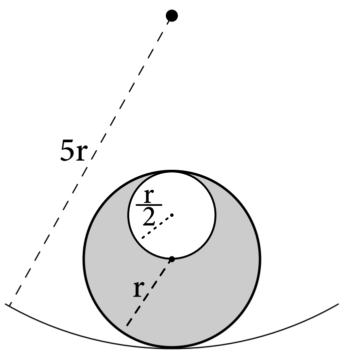
What is the moment of inertia of the onion about the axis of symmetry shown in the picture, if its mass is M?



38 An electric circuit consists of resistors with resistivity R, capacitors with capacity C, coils with inductance L, an ideal ammeter and ideal conductors. This time, the circuit has been connected to an AC-generator with peak voltage U and angular frequency ω. What is the peak current shown by the ammeter?

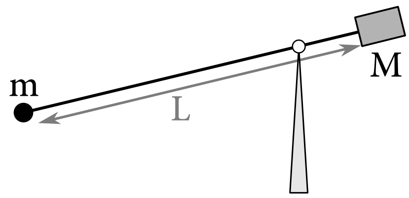


39 Jonas the coachman was riding on a bumpy road. One of his wheels with radius r got stuck in a pit with radius $5r$. While trying to pull the stage-coach out, the wheel broke off. On top of that, his forceful rescue attempts also resulted in a hole with radius $r/2$ as shown in the picture. Jonas was intrigued by the oscillating motion of the broken wheel in the pit. What is the period of its small oscillations?



40 George is a siege engineer. His next mission is to destroy a castle. On a trip to a Swedish furniture shop that shall not be named, he bought a magnificent TREBÖCHET – a massless rod of length L, a counterweight of mass M, a projectile of mass m and a pivot about which the rod was to rotate.

However, the trebuchet was delivered in parts and therefore he needs to assemble it. How far from the counterweight should he place the pivot, so that the linear acceleration of the projectile after releasing the counterweight is highest?



Solutions

1 All we need to do is to draw the net of the cuboid. The shortest path on the planar net is the shortest path between points A and B as well. However, there is more than one possible link as there are multiple possible nets. From the Pythagoras' theorem we can say that the length of the shortest link is either

$$s_1 = \sqrt{(a+b)^2 + c^2}, \quad s_2 = \sqrt{(b+c)^2 + a^2} \quad \text{or} \quad s_3 = \sqrt{(c+a)^2 + b^2}.$$

According to the condition from the task $a < b < c$ the shortest link is the one that includes the diagonal of the rectangle most similar to the square. In our case it is s_1 and thus the final time to reach the point B is

$$t = \frac{\sqrt{(a+b)^2 + c^2}}{v}.$$

2 The average velocity is calculated as a ratio of the total distance to the total time. We can obtain the total time from the graph. Total distance is calculated as the area under the curve of the graph, which is:

$$s = \frac{12 \cdot 10}{2} + \frac{(15-5) \cdot 4}{2} + 5 \cdot 4 + 10 \cdot 4 + \frac{(18-10) \cdot 6}{2} + 10 \cdot 6 + 18 \cdot 4 + \left(3^2 - \frac{\pi 3^2}{4}\right) + 15 \cdot 3 \doteq 342.931.$$

After substitution, it equals

$$v = \frac{s}{t} = \frac{342.931 \text{ m}}{33 \text{ s}} \doteq 10.39 \text{ m/s}.$$

3 First of all, we need to find out the average time the boys have to wait until a car picks them up. Then we add their travel time to Krakow. On average, every 200$^{\text{th}}$ car picks up a hitchhiker, so the average time the boys need to wait is $\frac{200}{k}$, where k is the frequency of the cars passing them.

The solution is then as follows: Michael arrives in Krakow in $\frac{200}{50} + 100 = 104$ minutes and Martin arrives in $\frac{200}{25} + 90 = 98$ minutes. The answer is then that Martin arrives in Krakow 6 minutes sooner.

4 It is necessary to realize what "percentage by volume" means. In our case, it is the ratio of the volume of ethanol to the volume of the whole drink. Percentage by volume can be converted to percentage by mass if we know the density of ethanol and the density of the drink. Percentage by mass ϱ then equals

$$\frac{V_{\text{et}} \rho_{\text{et}}}{V_{\text{drink}} \rho_{\text{drink}}} = \frac{m_{\text{et}}}{m_{\text{drink}}} = \varrho.$$

Density of ethanol is given, density of the drink can be calculated. The drink consists of water and ethanol. We know the percentage by mass of ethanol, the rest is water:

$$\rho_{\text{drink}} = 0.4 \rho_{\text{ethanol}} + 0.6 \rho_{\text{water}},$$

$$\rho_{\text{drink}} = 0.4 \cdot 790 \text{ kg/m}^3 + 0.6 \cdot 1000 \text{ kg/m}^3,$$

$$\rho_{\text{drink}} \doteq 916 \text{ kg/m}^3.$$

After substituting into the equation for percentage by mass we get:

$$\varrho = \frac{V_{\text{et}}\rho_{\text{et}}}{V_{\text{drink}}\rho_{\text{drink}}} = 0.4\frac{\rho_{\text{et}}}{\rho_{\text{drink}}},$$

$$\varrho = 0.4 \times \frac{790\ \text{kg/m}^3}{916\ \text{kg/m}^3},$$

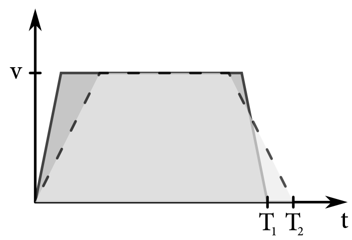
$$\varrho \doteq 0.345 = 34.5\ \%.$$

The drink contains aproximately 34.5 percent of alcohol by mass.

5 To calculate the total distance, we need to calculate the area under the curve of a $v(t)$ graph. According to our problem, the shape of this curve is a trapezoid. The area of the trapezoid is

$$s = \frac{v\,(a + c)}{2},$$

where a and c are the bases of the trapezoid and v is its height.



As we know, the total distance passed is the same in both cases, so are the areas of the trapezoids.

$$\frac{(2T_1 - t_1)\,v}{2} = \frac{(2T_2 - t_2)\,v}{2},$$

where T_1 is the total travel time on the journey there, T_2 is the total travel timeon the journey back and v is the speed limit, which, as we can see, cancels out. The time difference between the travel times is thus

$$T_1 - T_2 = \frac{t_1 - t_2}{2}.$$

6 We know that the water leaves the sprinkler with velocity v upwards. Each small element of water is then decelerated by the gravity. Time necessary to fall back on the ground can be derived from the kinematics

$$0 = vt - \frac{1}{2}gt^2.$$

Total time in the air is then

$$t = \frac{2v}{g}.$$

Now, we just have to calculate how much water flows through the nozzle during that time. According to the equation for mass flow $Q_m = \pi r^2 v \rho$, we can easily obtain the final mass of water in the air in any moment

$$m = \frac{2\pi r^2 v^2 \rho}{g}.$$

7 Due to Archimedes' principle, two forces affect our hollow brick in the air: gravitational and buoyant force. For the floating brick, the magnitude of these forces must be equal, thus

$$\rho_v V g = mg,$$

where ρ_v is density of air. The volume of the brick is $V = k \cdot 2k \cdot 4k = 8k^3$. The mass of the brick is calculated as its surface area multiplied by its surface density, thus

$$m = \sigma \cdot 2\left(k \cdot 2k + k \cdot 4k + 2k \cdot 4k\right) = 28k^2\sigma.$$

Having substituted the mass and volume into the first equation, we get

$$k = \frac{7\sigma}{2\rho_v} \doteq 10.77\ \mathrm{cm} \doteq 11\ \mathrm{cm}.$$

8 Let us assume that the layer of foam vanishes in time t. In the glass, the drink level rises by vt, while ut is the height of the glass filled with air that was originally trapped in foam; since the thickness of the foam layer was $(u + v)\,t$. Therefore, the volume fraction of air in the foam was $\frac{u}{u+v}$.

9 We can determine the centre of mass of a complicated body by splitting it into simpler shapes, then summing the products of their coordinates and masses and dividing the result by the total mass:

$$X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots + m_n x_n}{m_1 + m_2 + m_3 + \ldots + m_n},$$

$$Y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \ldots + m_n y_n}{m_1 + m_2 + m_3 + \ldots + m_n}.$$

Let's start with the x-coordinate of the centre of mass. We can use the fact than the center of mass of a triangle is located in the first third of its height. Using this information we obtain the equation

$$X = \frac{16m \cdot 10a + 2m \cdot 17.5a + 2 \cdot \frac{m}{2} \cdot \frac{17+18+18}{3}a + 3 \cdot 3m \cdot 19.5a + 3 \cdot \frac{m}{2} \cdot \frac{21+21+22}{3}a}{16m + 3m + 9m + \frac{3}{2}m} = \frac{2521}{177}a \doteq 14.242a.$$

With the y-coordinate, it is much easier, because the whole trident is symetric about the handle. Due to this fact, the y-coordinate is trivially

$$Y = 6.5a.$$

10 Poor hobbit. He jumped up with the hopes of surviving. Indeed, was he right? When jumping from ground, he reaches height h. Therefore, his muscles provide effective energy mgh. As the elevator cabin was much heavier than the hobbit, the amount of energy received by the cabin was negligible. The same happens when he jumps on the surface of the Earth.

Hobbit's impact velocity is thus equal to the difference between the speeds of the elevator and the jumping speed of the hobbit. From kinematic equations we get:

$$v_{\mathrm{impact}} = v_{\mathrm{lift}} - v_{\mathrm{jump}} = \sqrt{2gH} - \sqrt{2gh} = \sqrt{2g}\left(\sqrt{H} - \sqrt{h}\right),$$

where H is the height from which the lift was falling. Finally, we need to find the equivalent free-fall height. For motion with constant acceleration, the total distance is $s = \frac{v}{2g}$. For v we put v_{impact}, and we get

$$s = \frac{v^2}{2g} = \frac{1}{2g}\left(\sqrt{2g}\left(\sqrt{H} - \sqrt{h}\right)\right)^2 = \left(\sqrt{H} - \sqrt{h}\right)^2.$$

In the end, we plug in $h = 0.7$ m and $H = 3$ m. The result proves that the hobbit was right. His impact velocity was same as if he had jumped from just about 0.8 m.

11 We can solve this problem by comparing the initial and final energies of our system. Let us denote the height of the twin h, her density ρ_t and volume V. Let us set the reference level to the depth $h/2$ under the water.

A cylinder's center of mass is in its middle. When the twin emerges from the pool, her place will be taken by water. Its potential energy will decrease by

$$E_w = \rho_w V g \frac{h}{2}.$$

In the final state, only the twin possesses non-zero potential energy. If we denote the height of its center of mass above the reference level x, its value will be

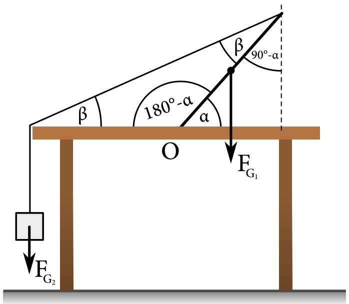
$$E_t = V g x \rho_t.$$

Comparing the energies tells us the height x

$$\rho_w V g \frac{h}{2} = V g x \rho_t \quad \Rightarrow \quad x = \frac{h\rho_w}{2\rho_t} = 4.5\ \mathrm{m},$$

which happens to be the same as the height of the upper base above the water level.

12 We want the system to be stationary. In equilibrium, the net force and torque in any particular direction is equal to zero. The net force is pretty easy to calculate from free body diagram (there are no reaction forces in the picture). It is obvious that it is torque what causes the rod to move. Therefore, we need the net torque acting on the object to be zero. Let's calculate torques about point O, about which reaction forces have no torque.



As the picture shows, the moment arm of rod's weight F_{G1} is $\frac{L}{2}\cos\alpha$, the moment arm of the mass's weight F_{G2} is $L\sin\beta$.

In order to keep the system balanced, the equality of torques is required

$$\tau_1 = \tau_2,$$

$$2mg\cos\alpha\frac{L}{2} = mgL\sin\beta,$$

$$\sin\beta = \cos\alpha,$$

Based on the given lengths, we know that the massless rope and the rod create the isosceles triangle. From the fact, that internal angles of a triangle sum to 180° we have

$$2\beta + (180° - \alpha) = 180°,$$

$$2\beta = \alpha$$

The torque equation about point O is then

$$\sin\frac{\alpha}{2} = \cos\alpha.$$

We either see the solution directly, or we can calculate it properly. The half-angle formula for sine $\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$ is utilized. After few steps we get

$$\cos^2\alpha + \frac{1}{2}\cos\alpha - \frac{1}{2} = 0,$$

it is the quadratic equation in cosine. Its solution is

$$\cos\alpha = -\frac{1}{4} \pm \frac{3}{4}.$$

We are interested in the plus sign solution because we expect the solution to be from interval $\left\langle 0\,;\,\frac{\pi}{2}\right\rangle$. It gives the angle

$$\alpha = \arccos\frac{1}{2} = 60°.$$

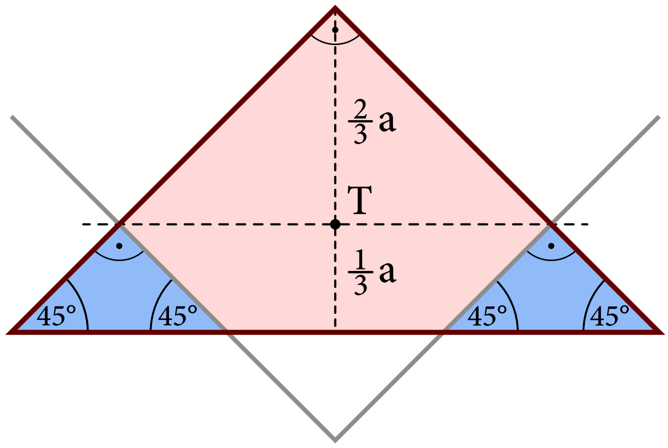
13 Current flowing through the ammeter can be calculated quite easily. Since ammeter is connected into main branch right next to the DC-generator, it is sufficient to calculate total resistance of the circuit. At first sight, it could seem to be complicated, but since it is only DC circuit, our problem could be simplified.

In DC circuit, capacitors behave like ideal insulators – current does not flow through them. Contrary, coils behave like ideal conductors – their resistance is zero. Therefore, entire circuit is equivalent to 3 parallel resistors with resistance R connected to DC-generator and ammeter. Hence, total resistance is $\frac{R}{3}$ and ammeter shows current

$$I = \frac{U}{\frac{R}{3}} = 3\frac{U}{R}.$$

14 The roof is affected by both weight and normal force. Its stability is ensured by total torque being equal to zero. Axis of rotation is determined by position of roof's resting points. Maximal protrusion goes with centre of gravity being positioned directly above the axis.

Remaining problem constitutes of total covered area calculation. Straightforward approach is to subtract the area of two smaller triangles formed by roof overlapping the walls from total area of roof $S_1 = \frac{a^2}{2}$, where a is length of cathetus. Brief thinking about geometry provides us with information that little triangles are right and isosceles as well as the large one. Next, we utilize that a median of a right triangle is also its altitude and an axis of angle opposite its hypotenuse. A centroid is located in one third of the median, closer to a side. That means that all small triangles are one-third in size.



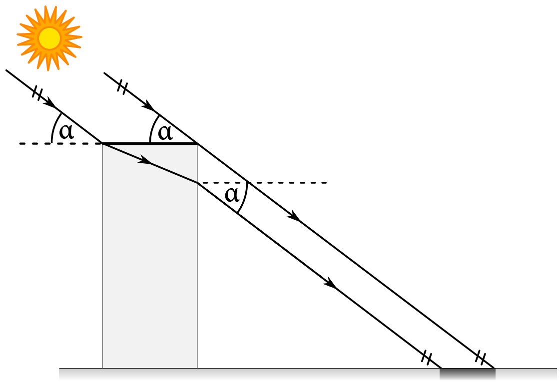
Therefore the surface of the roof without small triangles can be calculated as

$$S = \frac{a^2}{2} - 2 \cdot \left(\frac{1}{3}\right)^2 \cdot \frac{a^2}{2} = \frac{7}{18}a^2.$$

Hence the portion of the roof covering the summerhouse is

$$k = \frac{\frac{7}{18}a^2}{\frac{1}{2}a^2} = \frac{7}{9}.$$

15 In fact, this situation cannot happen except one particular case: the beam's angle of incidence and the angle of refraction are both α. We can really shorten the shadow using the prism, but it can never fully disappear, unless the light beams would be spreading horizontally with an angle 0°. Then the ground would not be illuminated at all and the base would not cast any shadow on the ground.



16 A common bubble contains the air that can be considered to be an ideal gas. Ideal gas law says that:

$$pV = NkT.$$

In our case k and N are constants, so the equation can be written as $\frac{pV}{T}$ = const.. Due to this fact, we can compare the bubble's final state near the surface with the state at the beginning

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}.$$

Temperatures T_1 and T_2 are given, volume V_1 can be calculated from the radius r_1 as $V_1 = \frac{4}{3}\pi r_1^3$. Radius r_2 is then equal to:

$$r_2 = \sqrt[3]{\frac{p_1 T_2}{T_1 p_2}} r_1.$$

Now we have to determine the pressures p_1 and p_2. It is obvious that the pressure on the top of the sea is $p_2 = p_a tm$. Due to hydrostatic pressure, the value of the p_1 has to be $p_1 = h\rho g + p_a$. Then:

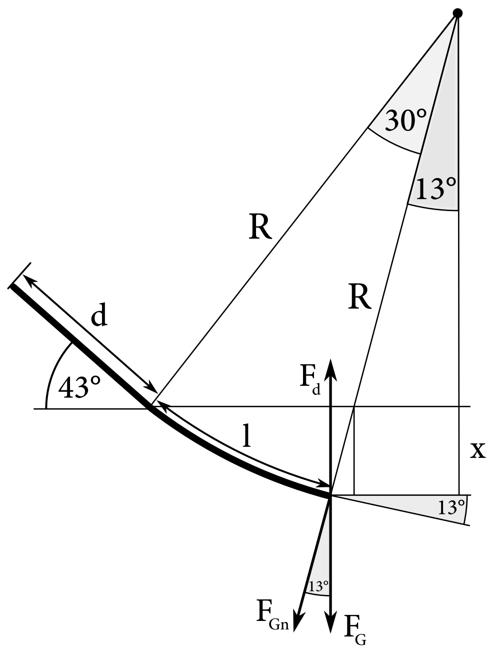
$$r_2 = \sqrt[3]{\frac{T_2}{T_1}\left(1 + \frac{h\rho g}{p_a}\right)} r_1.$$

After the substitution of the given values into the equation, we obtain a final result:

$$r_2 \doteq 9.5\ \text{cm}.$$

Let's notice we could have neglected surface tension, as the pressure caused by it is about five orders of magnitude smaller than atmospheric pressure.

17 First of all, why should Philip feel any g-force when he just freely moves down the hill? After a short brainstorming, we find out that in the final phase of movement he is on the curved hill, so there is centrifugal force F_d exerted on him. The magnitude of centrifugal acceleration is $a_d = \frac{v^2}{R}$, where R is radius of curvature of the end of jumping hill.



Let's take a look at the geometry of the jumping hill. There is vertical distance h between the highest and the lowest point of hill. That consists of an inclined plane of length $d = 50$ m and the change of height of the curved part, which is

$$h = d \sin 43° + x.$$

With a little bit of trigonometry, we can find out that

$$x = R\left(\cos 13° - \cos 43°\right).$$

We also know that on a circular arc of length $l = 50$ m, the slope tilt changes by 30°. Using this information, we can calculate radius of curvature of the hill as

$$R = \frac{6l}{\pi}.$$

We can get velocity at the moment of Philip's taking off from the law of conservation of energy. We know that $\frac{1}{2}v^2 = gh$. Then for centrifugal acceleration applies

$$a_d = \frac{v^2}{R} = \frac{2h}{R}g.$$

Using the geometry of the jumping hill, we get

$$a_d = \frac{2}{R}\left[d \sin 43° + R\left(\cos 13° - \cos 43°\right)\right] g = \left[\frac{\pi d}{3l} \sin 43° + 2\left(\cos 13° - \cos 43°\right)\right] g.$$

Let's not forget that Philip is in the Earth's gravity field. That means that also his weight contributes to the total g-force, specifically its normal component $F_{G\perp} = F_G \cos 13°$. For the total g-force, we get

$$a = a_d + g \cos 13° = \left(\frac{\pi d}{3l} \sin 43° + 3 \cos 13° - 2 \cos 43°\right) g \doteq 2.17\, g.$$

18 An elastic rope that is commonly used for bungee jumping behaves as a spring, therefore Adam' movement can be described as a movement of a harmonic oscilator. Period of a such oscilator is generally $T = 2\pi\sqrt{\frac{m}{k}}$. When Adam is hungry, his period on the rope is

$$T_1 = 2\pi\sqrt{\frac{M}{k}}.$$

Having eaten N pieces of pancakes, Adam's period changes to

$$T_2 = 2\pi\sqrt{\frac{M + Nm}{k}}.$$

From the task, we know that the new period should be $\frac{3}{2}T_1 = T_2$. Then we can say

$$\frac{3}{2}2\pi\sqrt{\frac{M}{k}} = 2\pi\sqrt{\frac{M + Nm}{k}},$$

After taking the derivation we obtain the final equation

$$N = \frac{5}{4}\frac{M}{m}.$$

19 The task is clear, we have to calculate resistance of individual circuits.

Firstly, we calculate resistance between vertices A and B. Resistance will not change if the top vertex of pyramid is replaced by ideal conductor which connects conductors from vertices A and B to other two. This ideal conductor lies at the axis of reflection symmetry of the circuit. Therefore after connecting voltage generator to circuit, both ends of ideal conductor have identical potential. That means, no current flows through it and this identical conductor could be removed from circuit. Equivalent circuit is shown below.

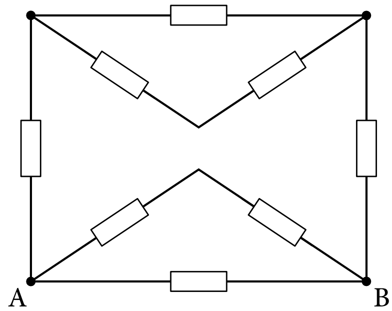


Figure 1: *Equivalent circuit between vertices A and B*

Resistance of every conductor is replaced by resistor with resistance $R = \xi L$ and ideal conductors. We can easily see, that it is ordinary series and parallel circuit, hence total resistance is

$$R_{\mathrm{AB}} = \frac{\left[2R + \frac{2R \cdot R}{2R+R}\right]\frac{2R \cdot R}{2R+R}}{2R + \frac{2R \cdot R}{2R+R} + \frac{2R \cdot R}{2R+R}} = \frac{8}{15}R.$$

An identical trick (removing of conductor with no current flowing through it) could be also used for calculation of resistance between vertices A and C. This time, two conductors lies on the axis of symmetry. If we remove them, we will get equivalent circuit

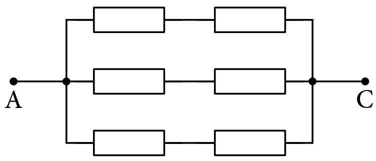


Figure 2: *Equivalent circuit between vertices A and C*

It is easy to calculate final resistance

$$R_{AC} = \frac{2}{3}R.$$

The sought ratio of resistances is

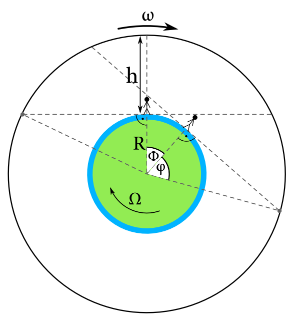
$$\frac{R_{AB}}{R_{AC}} = \frac{\frac{8}{15}R}{\frac{2}{3}R} = \frac{4}{5}.$$

20 Let's assume that the flying satellite's height above the surface is h and its trajectory is circular. It is held on its trajectory by gravity in the role of centrifugal force, so we can write

$$G\frac{Mm}{(R+h)^2} = m\omega^2 (R+h),$$

where R is the Earth's radius, M is the Earth's mass, m is the satellite's mass and ω is the satellite's orbital angular velocity. Then

$$\omega = \sqrt{\frac{GM}{(R+h)^3}}.$$



Let the time of the overflight be T. Over this time it changes its position by angle $\phi = \omega T$. The angle determined by horizons and Earth centre is $\varphi = 2\arccos\frac{R}{R+h}$. Let's not forget that due to the Earth's rotation Martin's position changes by angle $\Phi = \Omega T$, where Ω is the Earth's angular velocity. After putting it in equality we get

$$\sqrt{\frac{GM}{(R+h)^3}}T = \Omega T + 2\arccos\frac{R}{R+h},$$

from which we would like to determine height h.

Here comes the problem – we got the equation we cannot solve analytically. Hence, we will solve it numerically using the binary search algorithm. For this purpose let's define function

$$F(h) = \sqrt{\frac{GM}{(R+h)^3}}T - \Omega T - 2\arccos\frac{R}{R+h}$$

and we will search for its zero point $F(h) = 0$.

The principle of binary search is the following: First of all we estimate the interval $(a; b)$, where we suppose we find root of the equation.

Due to the fact, that the function $F(h)$ is continuous, if $F(a) \cdot F(b) < 0$, i. e. if the signs at the end points of the interval are opposite, then $\exists c \in (a; b) : F(c) = 0$.[1]

Subsequently we split the interval in two halves, which gives us two intervals $\left(a; \frac{a+b}{2}\right)$ and $\left(\frac{a+b}{2}; b\right)$.

If we are lucky and get $F\left(\frac{a+b}{2}\right) = 0$, we finish the algorithm. Otherwise we continue searching in one of two intervals – the one that has opossite signs at its end points. We finish the algorithm after getting the required accuracy. In our case, we require accuracy 1 km, which means that we finish the algorithm when the length of the interval gets smaller than this value.

Now, we are interested in height of the satellite above surface. It is seemingly smaller than 1000 km, so as our initial estimation we can use interval (0 km; 1000 km). Our goal is to reach the accuracy 1 km. Due to the fact, that after every iteration the interval's length is reduced by half, we need at least 10 iterations, which gives us the interval size $\frac{1000\ \text{km}}{2^{10}} < 1$ km. We can write down the whole calculation into the table.

Iteration	a [m]	b [m]	$\frac{a+b}{2}$ [m]	$F(a)$	$F(b)$	$F\left(\frac{a+b}{2}\right)$
0	0	1 000 000	500 000	0.56	−0.61	−0.27
1	0	500 000	250 000	0.56	−0.27	−0.02
⋮			⋮			⋮
10	230 469	231 445	230 957	0.000 48	−0.000 73	−0.000 13

In the tenth iteration, we finally got $b - a < 1$ km, so we reached the required accuracy. As our estimating numerical solution, we can use the value from the middle of the interval $h = 231.0$ km. If we continued with iterations, we would get the accurate solution $h \doteq 230.9$ km.

21 Before we start drawing the VT–diagram, we need to find out what exactly happens with temperature and volume during the cyclical process. The only thing we need is to use the ideal gas law $pV = NkT$, where N is non-changing number of particles.

- During the transition 1 → 2 pressure rises proportionally to volume which means that based on the ideal gas equation $\frac{pV}{T}$ = konšt., we know that volume and temperature change according by $V \propto \sqrt{T}$. Thus, the gas has volume $2V_0$ and temperature $4T_0$ in point 2.
- The transition 2 → 3 is an isobaric process which means that volume changes proportionally to temperature. Then the gas has volume $4V_0$ and temperature $8T_0$ in point 3.

[1] In addition, the function $-\arccos\frac{R}{R+h}$ is monotonically increasing over the interval $h \in [0; \infty]$ and similarly $\sqrt{\frac{GM}{(R+h)^3}}$ is monotonically decreasing over the interval, so the equation $F(c) = 0$ has just one solution.

- During the transition 3 → 4, pressure decreases proportionally with volume. The gas has volume $2V_0$ and temperature $2T_0$ in point 4.
- During the transition 4 → 1, volume changes linearly with the temperature as well.

While drawing the VT–diagram we need to make sure that all the curves cross the origin of the coordinate system after their prolonging. The cycle in VT–diagram is then:

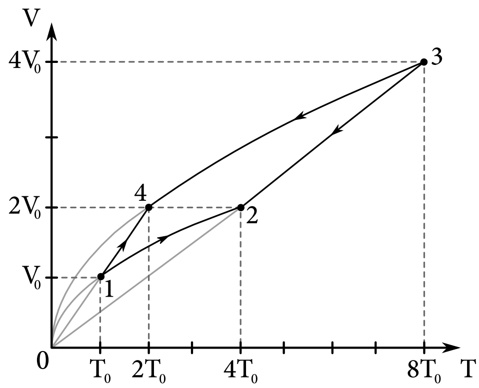


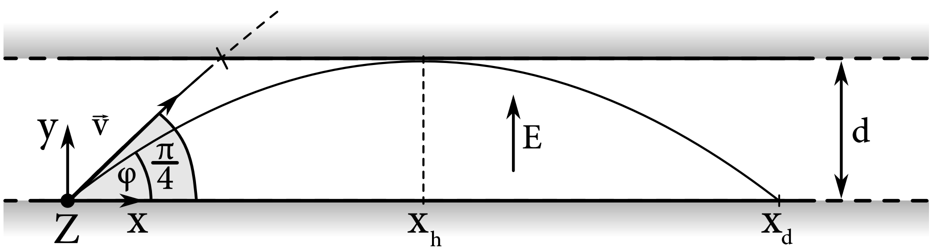
Figure 3: *The cycle redrawn to VT-diagram*

22 Since the electrons move through a homogeneous electric field, a constant force Ee will act upon them. This force will impart constant downward acceleration $\frac{Ee}{m_e}$ to each electron. We can note that this acceleration is much greater than the gravitational acceleration, therefore the later can be neglected. As the whole setup is rotationally invariant, we will analyze only electrons moving in a single plane containing the source.

Similarly to motion in a homogeneous gravitational field, we can describe the x and y coordinates of an electron launched at an angle φ as

$$x = v_x t = v \cos\varphi \cdot t,$$

$$y = v_y t - \frac{1}{2}at^2 = v \sin\varphi \cdot t - \frac{1}{2}at^2.$$



Generally, three following situations must be distinguished depending on electron's initial velocity and their angle of elevation:

- electron hits the upper plate of capacitor;
- electron just skims the upper plate of capacitor;
- electron does not reach a sufficient height for touching the upper plate of capacitor.

Let us look at the electron which will just skim the upper plate. A flying electron reaches its maximum height after time T

$$T = \frac{v_y}{a} = \frac{v \sin\varphi}{a}.$$

Substituting this time into the equation for the y-coordinate and setting it equal to d tells us the angle under which an electron has to be launched in order to reach its maximum height right at the upper plate

$$d = \frac{v^2 \sin^2\varphi}{a} - \frac{v^2 \sin^2\varphi}{2a} = \frac{v^2 \sin^2\varphi}{2a},$$

$$\varphi = \arcsin\frac{\sqrt{2ad}}{v}.$$

If we substitute in the given values, we obtain the value $\varphi \doteq 36.37°$. This is less than 45°, which means that any electron which could potentially reach further distance hits the upper plate. To find how far the electrons fly before they skim the plate, we substitute this angle together with time T into the equation for the x-coordinate

$$x_h = \frac{v^2 \cos\varphi \sin\varphi}{a}.$$

If we remark on the fact that electrons attain their maximum height at one half of the total horizontal distance traveled, it comes as no surprise that the maximum reach of electrons on the bottom plate is $x_d = 2x_h$.

After entering the given values we get $x_h = 2.715$ m, which means that electrons on the upper plate will be hitting a circle of radius 2.715 m and a circle of radius 5.43 m on the bottom plate. The whole area hit by the electrons is

$$S = \pi\left(x_h^2 + x_d^2\right) = 5\pi x_h^2 \doteq 116\ \mathrm{m}^2.$$

23 Let us analyze the force acting on the table. It has to be the gravity of the bowl $F_{\mathrm{bowl}} = Mg$ as well as the gravity of the water $F_{\mathrm{water}} = \frac{2}{3}\pi R^3 \rho g$.

Now, let us look at the force pushing the bowl up. It is essential to consider pressure of the water. Following the third Newton law, the same force as is acting on the table has to act from the table on the water. It is this pressure $p = h\rho g$ that causes the uplifting of the bowl. Which gives us the force $F = pS = \rho g \pi R^3$.

Now we just compare the two forces in an equation and derive the mass of the bowl:

$$\rho g \pi R^3 = Mg + \frac{2}{3}\pi R^3 \rho g,$$

from which

$$M = \frac{1}{3}\pi R^3 \rho.$$

24 Let's determine the torques about a point that connects the barrel with the hill. Two different torques must be considered.

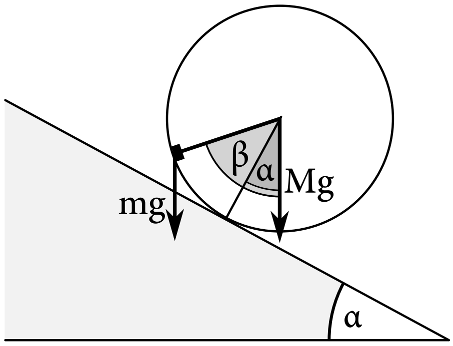


Figure 4: *Barrel with cheese inside*

The torque due to gravity of the barrel rotates the barrel clockwise, the torque due to gravity of the cheese rotates the barrel anticlockwise. The torques have following magnitudes:

$$M_{\text{cheese}} = mgR\left(\sin\beta - \sin\alpha\right),$$

$$M_{\text{barrel}} = -MgR\sin\alpha.$$

In case of equillibrium, the sum of all torques has to be zero, thus:

$$M_{\text{cheese}} + M_{\text{barrel}} = 0,$$

$$mgR\left(\sin\beta - \sin\alpha\right) = MgR\sin\alpha,$$

and then

$$\frac{m}{M+m}\sin\beta = \sin\alpha.$$

To maximize the value of α, we have to maximize $\sin\alpha$ and then evaluate it. The $\sin\alpha$ is proportional to $\sin\beta$ and due to this fact we have to maximize even the $”βangle. Therefore, the equality between tangential component of gravity and friction force has to be satisfied in order to reach the stable state of the cheese in the barrel:

$$mg\sin\beta = mg\cos\beta\mu,$$

$$\tan\beta = \mu.$$

According to formula $\cos\beta = \sqrt{1-\sin^2\beta}$, we obtain equation

$$\sin\beta = \frac{\mu}{\sqrt{1+\mu^2}}.$$

After subtitution

$$\sin\alpha = \frac{m}{M+m}\frac{\mu}{\sqrt{1+\mu^2}}$$

And finally:

$$\alpha = \arcsin\left(\frac{m}{M+m}\frac{\mu}{\sqrt{1+\mu^2}}\right).$$

25 We know that if we fill a whole capacitor with a dielectric with relative permittivity ε, its capacitance will increase ε-times. But what if we fill only a part of it? One option is to find an equivalent scheme which consists only of capacitors which are either completely filled or empty.

Imagine that we insert a thin metal board parallel with the capacitor's plates into it. This does not change the electric fields inside the capacitor, since the total charge on the board is zero. That means that electrical properties of a such capacitor are same as before.

If we now cut the board parallel with its surface into two boards and connect them with a conductor, we effectively obtain two series-connected capacitors. Don't forget that the new capacitors' capacitance will be doubled, due to the halved distance between their plates.

We find out that we can replace the first capacitor with an equivalent of two capacitors connected in series, each with capacitance $2C$, and only one of them filled with the dielectric:

$$C_1 = \frac{2C \cdot \varepsilon 2C}{2C + \varepsilon 2C} = \frac{2\varepsilon}{1+\varepsilon} C.$$

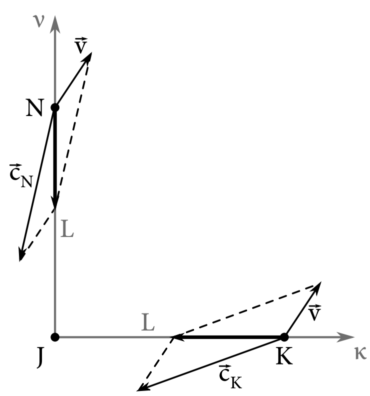
In the second case, it is obvious that if we cut the capacitor vertically, perpendicularly to its plates, the fields again does not change and we get two capacitors connected in parallel. This time, the new capacitances will be halved, due to halved area of the plates. Thus we can express the capacitance of the second capacitor as

$$C_2 = \frac{C}{2} + \varepsilon \frac{C}{2} = \frac{1+\varepsilon}{2} C.$$

And finally, their ratio is

$$\frac{C_1}{C_2} = \frac{4\varepsilon}{(1+\varepsilon)^2}.$$

26 Firstly, let's probe the first phase, in which Nina and Martin shout and Oliver listens to them. The sound propagates as spherical wavefronts with respect to the air. Its movement can be described by an unknown constant vector $\vec{v}$. This vector is composed with a vector $\vec{c}$ denoting the propagation of soundwave in the air, so that the resulting vector points towards Oliver.



Denote a velocity of propagation of Martin's shout as $\vec{c_K}$. The vector was chosen in a such way that it fulfills

$$c_{K\nu} + v_\nu = 0 \qquad \text{a} \qquad c_{K\kappa} + v_\kappa = -\frac{L}{T_{KJ}},$$

L being a distance of both Nina and Martin from Oliver and T_{KJ} be given time which is needed for the sound to propagate from Martin to Oliver. Analogously, a vector $\vec{c_N}$ fulfills

$$c_{N\kappa} + v_\kappa = 0 \qquad \text{and} \qquad c_{N\nu} + v_\nu = -\frac{L}{T_{NJ}},$$

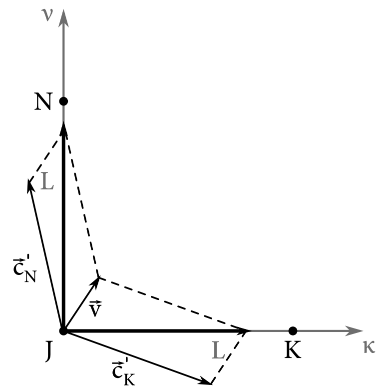
and naturally both vector must be equal to each other

$$\left|\vec{c_K}\right| = \left|\vec{c_N}\right| = c.$$

The expressions above allow us to eliminate $\vec{c_K}$ and $\vec{c_N}$ from further calculations.

$$c^2 = \left(\frac{L}{T_{KJ}} + v_\kappa\right)^2 + v_\nu^2 = \left(\frac{L}{T_{NJ}} + v_\nu\right)^2 + v_\kappa^2$$

The situation turns around in the second phase. The sound propagates from Oliver to Martin and Nina in directions $\vec{c_K}' + \vec{v}$ and $\vec{c_N}' + \vec{v}$ respectively. They must follow analogous relations to the ones in the previous phase.



Therefore

$$c'_{K\nu} + v_\nu = 0, \qquad c'_{K\kappa} + v_\kappa = \frac{L}{T_{JK}},$$

$$c'_{N\kappa} + v_\kappa = 0, \qquad c'_{N\nu} + v_\nu = \frac{L}{T_{JN}}$$

and

$$\left|\vec{c_K}'\right| = \left|\vec{c_N}'\right| = c.$$

Having eliminated $\vec{c_K}'$ and $\vec{c_N}'$ from the set of equations, one obtains

$$c^2 = \left(\frac{L}{T_{KJ}} - v_\kappa\right)^2 + v_\nu^2 = \left(\frac{L}{T_{NJ}} - v_\nu\right)^2 + v_\kappa^2.$$

Furthemore, having combined it with the other expression for c^2, one gets

$$v_\kappa^2 + v_\nu^2 + 2\left(\frac{L}{T_{KJ}}\right)v_\kappa + \left(\frac{L}{T_{KJ}}\right)^2 =$$
$$= v_\kappa^2 + v_\nu^2 + 2\left(\frac{L}{T_{NJ}}\right)v_\nu + \left(\frac{L}{T_{NJ}}\right)^2 =$$
$$= v_\kappa^2 + v_\nu^2 - 2\left(\frac{L}{T_{JK}}\right)v_\kappa + \left(\frac{L}{T_{JK}}\right)^2 =$$
$$= v_\kappa^2 + v_\nu^2 - 2\left(\frac{L}{T_{JN}}\right)v_\nu + \left(\frac{L}{T_{JN}}\right)^2 .$$

This set of equations can be simplified by subtracting of the first two terms of each part of equation.

$$2\left(\frac{L}{T_{KJ}}\right)v_\kappa + \left(\frac{L}{T_{KJ}}\right)^2 =$$
$$= 2\left(\frac{L}{T_{NJ}}\right)v_\nu + \left(\frac{L}{T_{NJ}}\right)^2 =$$
$$= -2\left(\frac{L}{T_{JK}}\right)v_\kappa + \left(\frac{L}{T_{JK}}\right)^2 =$$
$$= -2\left(\frac{L}{T_{JN}}\right)v_\nu + \left(\frac{L}{T_{JN}}\right)^2 .$$

An equality of the first and the third row, and the second and the fourth row respectively leads to

$$v_\kappa = \frac{L}{2}\left(\frac{1}{T_{JK}} - \frac{1}{T_{KJ}}\right) \qquad \text{and} \qquad v_\nu = \frac{L}{2}\left(\frac{1}{T_{JN}} - \frac{1}{T_{NJ}}\right),$$

which gives

$$\left|\vec{v}\right| = \frac{L}{2}\sqrt{\left(\frac{1}{T_{JK}} - \frac{1}{T_{KJ}}\right)^2 + \left(\frac{1}{T_{JN}} - \frac{1}{T_{NJ}}\right)^2} = 29.43\ \mathrm{m/s}.$$

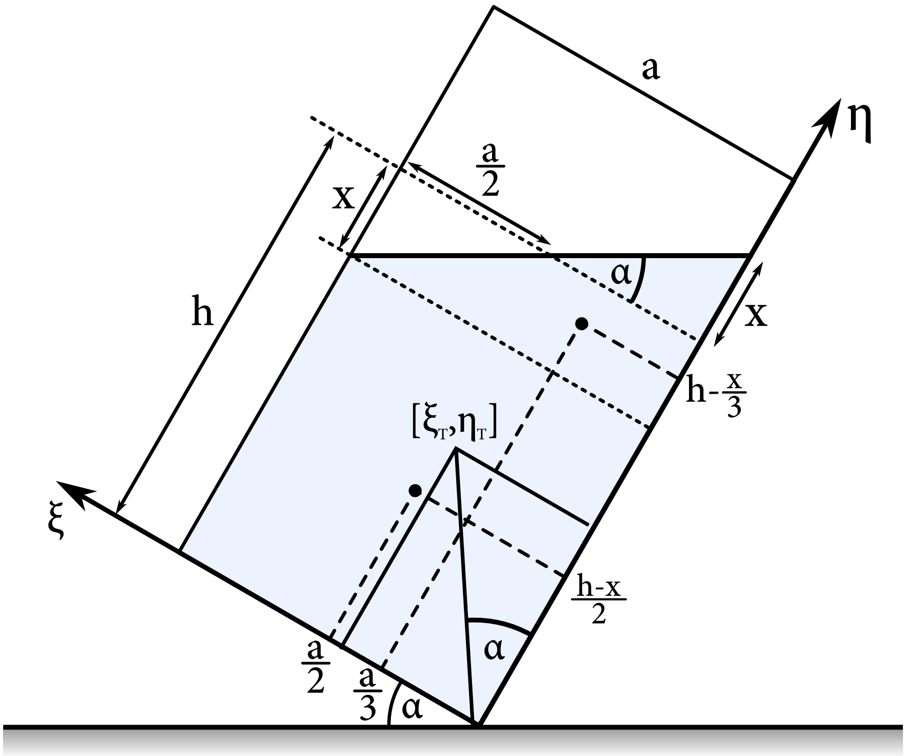
27 Total magnetic field can be determined by superposition of magnetic fields of the individual edges of the original square. However, who would want to divide squares into pieces if we could put final circuit together from individual squares?

It is enough to take three original circuits and place them on designated places, so that they create final formation. In our case, it is the one square on the left side, the second one on the right side and the third one on the top side. Because of superposition of currents, there is no flow through two edges of the top base.

Magnetic field is perpendicular to original square, therefore fields from left and right side cancel out. Resultant magnetic field points from the top base to the centre, and its magnitude is same as magnitude of the original one. In other words, $B_2 = B_1$.

28 Let's assume that the surface of the milk is in the high h. If we place the milk on the inclined plane with the inclination of α, the milk surface will by still horizontal.

According to the problem, there is a reasonable amount of milk in the box. It will take the shape of a truncated prism. Its side face will have the shape of a trapezium (picture).



In the limiting case, the centre of mass of the milk has to be above the axis of rotation. Let's find the center of mass of the system. Firstly, we define coordinate system axes of which coincide with the edges of the box.

The trapezium consists of the rectangle with size $a \times (h - x)$ and a righti-angled triangle, whose perpendicular sides have sizes a and $2x$. The coordinates of the center of mass of the rectangle are $\left[\frac{a}{2}; \frac{h-x}{2}\right]$, The coordintaes of the center of mas of the right triangle are $\left[\frac{a}{3}; h - \frac{x}{3}\right]$. Therefore, the coordinates of the center of mass of the trapezium are

$$\xi_T = \frac{\frac{a}{2} \cdot a\,(h-x) + \frac{a}{3} \cdot ax}{ah} = \frac{a}{2} - \frac{ax}{6h};$$

$$\eta_T = \frac{\frac{h-x}{2} \cdot a\,(h-x) + \left(h - \frac{x}{3}\right) \cdot ax}{ah} = \frac{h}{2} + \frac{x^2}{6h}.$$

It can be easily seen that the inclination of the liquid's surface is

$$\tan\alpha = \frac{2x}{a}.$$

Moreover, from the condition that in the boundary case the center of mass of the milk has to be above the edge of the box we get

$$\tan\alpha = \frac{\xi_T}{\eta_T} = \frac{12ah - 2a^2 \tan\alpha}{12h^2 + a^2 \tan^2\alpha}.$$

We eliminate the unknown x and we get the quadratic equation for the unknown hight h of the milk in the box.

$$h^2 - \frac{a}{\tan\alpha}h + a^2\left(\frac{\tan^2\alpha}{12} + \frac{1}{6}\right) = 0.$$

Its solution is

$$h = \frac{a}{2\tan\alpha}\left(1 \pm \sqrt{1 - \tan^2\alpha\left(\frac{\tan^2\alpha}{3} + \frac{2}{3}\right)}\right).$$

Which result are we to take? The one with the plus "+" sign! The second solution (with minus sign) will decrease if we increase the angle of inclination. It would mean that the smaller angle we have the less milk is required for the tipping of the box. In the terminal case, the minus sign solution says that the box would tip over even when there would be no milk in the box. It is wrong of course!

For given numeric values of quantities $a = 1$ dm and $\alpha = 30°$ we get $h \doteq 1.6$ dm and therefore the volume of the milk in the box is

$$V = a^2 h \doteq 1.6\ \mathrm{dm}^3.$$

29 Before its break-up, the poor planet had been orbiting with a circular velocity at a distance r

$$v_c = \sqrt{\frac{GM}{r}}.$$

After the break-up, both pieces were travelling on a parabolic trajectory, which means they were moving exactly with the escape velocity. Its value at the distance r is

$$v_e = \sqrt{\frac{2GM}{r}}.$$

That means the velocity of the first piece changed by

$$\Delta v_1 = v_e - v_k = \left(\sqrt{2} - 1\right)\sqrt{\frac{GM}{r}}$$

and the velocity of the second one by

$$\Delta v_2 = v_e + v_k = \left(\sqrt{2} + 1\right)\sqrt{\frac{GM}{r}}.$$

The law of conservation of momentum provides us with a constraint on the masses and velocities

$$\Delta v_1 m_1 = \Delta v_2 m_2,$$

and so

$$\frac{m_2}{m_1} = \frac{\Delta v_1}{\Delta v_2} = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} = 3 - 2\sqrt{2}.$$

30 Having connected the ring to voltage U, a current starts flowing through the circuit. As we know, the current takes the path of less resistance. However, as the length of the wire increases, the resistance increases as well. It corresponds to the formula $R = \rho\frac{l}{S}$, where S is the area of the cross section of the wire, ρ is resistivity and l is length of the wire. Based on this information, we know the shorter the arc of the ring is, the larger current flows through it. Therefore

$$I \propto \frac{1}{R} \quad \Rightarrow \quad I \propto \frac{1}{l}.$$

The magnetic field around a conductor is proportional to the current and effective length of the conductor. We can imagine bend arc as really short conductors in the same distance from the center. Both of them contribute to the net magnetic field by same weight. Be aware! The direction of the magnetic field is important as well. It can be easily determined by the Amper's right hand rule. Therefore

$$B_{\text{up}} \propto I_1 \cdot l_1 \quad \text{a} \quad B_{\text{down}} \propto I_2 \cdot l_2.$$

What does it mean? While the current decreases with increasing length, the magnitude of the magnetic field increases. From the fact, that the net magnetic field depends linearly on I and l, these two quantities cancel out and the net magnitude of the magnetic field in the center does not depend on the angle at all.

For each component of the net magnetic field, we get

$$B_{\text{up}} = B_{\text{down}}.$$

However, the first one is directed upside, the second one down. Therefore, the net magnitude of the magnetic field is zero, no matter where the points of connection are.

31 Let's assume that the pendulum is rotating with angular velocity ω. Then it performs one rotation in time $T_o = 2\pi/\omega$. An equilibrium corrsponding to this angular velocity is deflected from the vertical direction by an angle φ_0. Two forces determine this deflection – centrifugal force $F_C = m\omega^2 R \sin\varphi_0$ and gravity $F_G = mg$. The deflection is described by

$$\tan\varphi_0 = \frac{F_C}{F_G} = \frac{\omega^2 R \sin\varphi_0}{g},$$

and from there

$$\cos\varphi_0 = \frac{g}{\omega^2 R}; \qquad \sin\varphi_0 = \sqrt{1 - \left(\frac{g}{\omega^2 R}\right)^2}.$$

Let's write down an equation of motion of the pendulum. Denote the angle measured from equilibrium as φ. Then the equation of motion takes the form

$$mR^2\varepsilon = -mgR\sin\left(\varphi_0 + \varphi\right) + m\omega^2 R\sin\left(\varphi_0 + \varphi\right) R\cos\left(\varphi_0 + \varphi\right).$$

It can be simplified utilizing sum formulae for goniometric functions

$$\varepsilon = -\frac{g}{R}\left(\sin\varphi_0\cos\varphi + \sin\varphi\cos\varphi_0\right) + \omega^2\left(\sin\varphi_0\cos\varphi + \sin\varphi\cos\varphi_0\right)\left(\cos\varphi_0\cos\varphi - \sin\varphi_0\sin\varphi\right).$$

This equation is not tempting at all, therefore we attempt to linearize it. We can use following approximations for small angles: $\sin\varphi \approx \varphi$ and $\cos\varphi \approx 1$. Thus

$$\begin{aligned}\varepsilon &\approx -\frac{g}{R}\left(\sin\varphi_0\cdot 1+\varphi\cdot\cos\varphi_0\right)+\omega^2\left(\sin\varphi_0\cdot 1+\varphi\cdot\cos\varphi_0\right)\left(\cos\varphi_0\cdot 1-\sin\varphi_0\cdot\varphi\right)\\ &\approx \left(\omega^2\sin\varphi_0\cos\varphi_0-\frac{g}{R}\sin\varphi_0\right)+\left(\omega^2\cos^2\varphi_0-\omega^2\sin^2\varphi_0-\frac{g}{R}\cos\varphi_0\right)\varphi.\end{aligned}$$

Recall that we have already found a relation between angle φ_0 and angular velocity ω and the length of the pendulum R. Having substituted the expresions for sines and cosines into the equation of motion, the constant term drops out and we obtain the equation of a harmonic oscilator

$$\varepsilon \approx \left(\frac{g^2}{\omega^2 R^2}-\omega^2\right)\varphi,$$

a period of small oscilations of which is

$$T=\frac{2\pi}{\sqrt{\omega^2-\frac{g^2}{\omega^2R^2}}}.$$

Therefore the sought ratio of the periods is

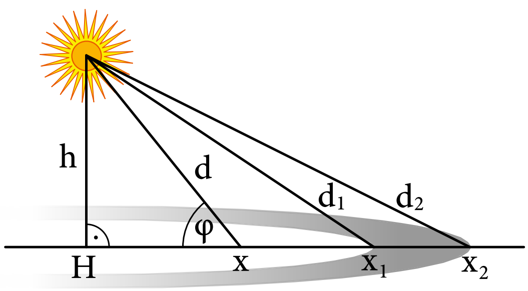
$$k=\frac{T_o}{T}=\sqrt{1-\left(\frac{g}{\omega^2 R}\right)^2},$$

which can be expressed in terms of equilibrium angle as

$$k=\sin\varphi_0.$$

32 Our job is to find the thermal equilibrium between two radiating black bodies. The Sun radiates isotropically, equal power in all directions. As Crêpearth absorbs this radiation, it eventually reaches some temperature and radiate on its own. In the equilibrium, both the absorbed and radiated power have to be equal, so we have an equilibrium condition to look for. Let us first draw a diagram with all the relevant quantities:

The flux density is inversely proportional to the square of the distance to the source. Therefore, it is clear that the greatest temperature will be attained directly below the sun, at a point which we have marked with H. Let us denote the temperature in this point as T_H. Every other point on Crêpearth is located further away from the Sun, furthermore, the rays come in at an angle, which means the same flux is distributed across a larger area. So we can say that for large distances the temperature will decrease and approach zero.



Let us denote the boundaries of our desired temperature interval as T_1 and T_2, with $T_1 < T_2$. There are three options for the possible shape of our habitable zone:

- If $T_H < T_1$, the whole Crêpearth will be too cold to sustain any habitable zone.
- If $T_1 < T_H < T_2$, the habitable zone will form a disk around the point H.
- And finally, if $T_2 < T_H$, the point H becomes too hot, but since the temperature decreases with distance, our habitable zone will shift away from the middle, and form an annulus around it.

So let's start with the calculations. We can easily determine the total power radiated by the Sun by using the Stefan-Boltzmann law:

$$P = S_\odot \sigma T_\odot^4 = 4\pi R_\odot^2 \sigma T_\odot^4,$$

where $S_\odot$ is the Sun's surface area. What we would like to know next is what fraction of this power hits a small area located at distance d. As the Sun is located sufficiently far from Crêpearth, we can consider it a point source. Since the whole power distributes across a sphere, the flux at a distance d is

$$\Phi = \frac{4\pi R^2 \sigma T_\odot^4}{4\pi d^2} = \frac{\sigma R^2 T_\odot^4}{d^2}.$$

Furthermore, since the flux reaches our surface at an angle φ, we need to multiply our expression by $\sin\varphi$. But we can express that by using known quantities:

$$\sin\varphi = \frac{h}{d}.$$

So we have worked how much of the power reaches a small surface of Crêpearth located at distance d. Now for our equilibrium condition we know that the radiated power has to be the same as the absorbed power

$$\frac{\sigma R^2 T_\odot^4 h}{d^3} \overset{!}{=} \sigma T^4.$$

We can express the distance d as

$$d = \left(R^2 h \frac{T_\odot^4}{T^4}\right)^{1/3}.$$

This is the direct distance to the Sun, so we can translate between it and the distance to the point H, denoted x, by using the Pythagorean theorem

$$x^2 = d^2 - h^2.$$

If we substitute the given values, we find two real solutions, which means that the third option occurs – our zone has the shape of an annulus. Finally, we can calculate its surface area as the difference between two circles

$$S = \pi\left(x_2^2 - x_1^2\right) = \pi\left(\left(R^2 h \frac{T_\odot^4}{T_1^4}\right)^{2/3} - \left(R^2 h \frac{T_\odot^4}{T_2^4}\right)^{2/3}\right) = \pi R^{4/3} h^{2/3} T_\odot^{8/3}\left(T_1^{-8/3} - T_2^{-8/3}\right).$$

Substituting the values gives us the result

$$S \doteq 3\,538\,765\ \mathrm{km}^2.$$

33 It is obvious that the astronaut will move along an ellipse with its focus in the middle of the asteroid. The best thing we can do right away is to draw a picture:

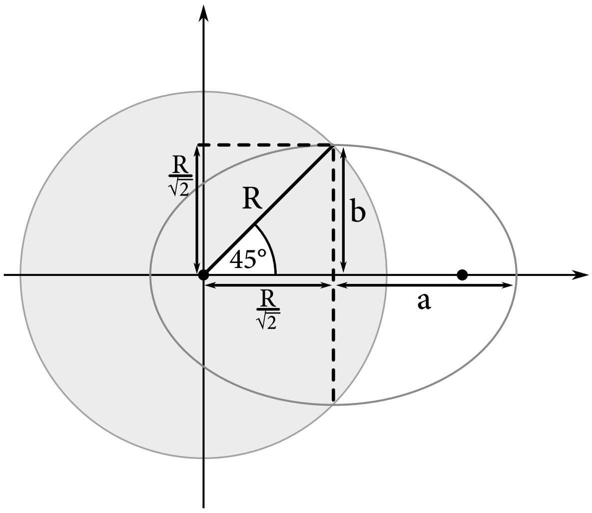


Figure 5: *Geometry of an elliptical orbit*

As according to our problem's premises, the launch and landing points are separated by one quarter of the circumference and thus the trajectory forms an angle of 45° with the planet's surface. The ellipse's semi-minor axis becomes immediately clear: $b = \frac{R}{\sqrt{2}}$. It is also clear that the astronaut's motion will run along exactly half of the ellipse.

In order to find the semi-major axis, we need to use a property that the sum of distances to two focal points is same for any point on the ellipse. If we pick the top point in our picture, where the astronaut originally stood, we get $2R$ to be the sum of distances to the foci and $2a$ if we pick the leftmost point, the pericenter. This means that $a = R$. Having determined the ellipse, we can now move on to the time. We will start with the general formula for the orbital period around the entire ellipse[2]

$$T = \frac{2\pi}{\sqrt{GM}} a^{\frac{3}{2}}.$$

The next thing we need is the second Kepler's law, which says that a line connecting the focus and the orbiting body always sweeps an equal area in a certain amount of time. That means the time needed to move along half of the ellipse is in the same proportion to the total time as the swept-out area to the total area. The swept-out area is

$$s = \frac{\pi ab}{2} + \frac{R^2}{2},$$

the total area is

$$S = \pi ab,$$

therefore

$$t = T\frac{s}{S} = \frac{2\pi}{\sqrt{GM}} a^{3/2} \frac{\frac{\pi ab}{2} + \frac{R^2}{2}}{\pi ab} = \frac{\left(\pi + \sqrt{2}\right) R^{3/2}}{\sqrt{GM}}.$$

[2]In case you don't remember this, it can be derived by finding the period around a circle and then replacing the radius with semi-major axis, which can be justified by the third Kepler's law.

34 Why should Martin see the red light as green? After short brainstorming, we can figure out it can only be caused by Doppler's effect. But there is a catch! The light travels with the speed of light (surprisingly). Due to this fact we are forced to use the relativistic Doppler effect. Then the observed frequency can be calculated using:

$$f = f_0\sqrt{\frac{c+v}{c-v}}.$$

If Martin should have seen the red light as green, the velocity of his car would have been

$$v = \frac{f^2 - f_0^2}{f^2 + f_0^2}c = \frac{\lambda_0^2 - \lambda^2}{\lambda_0^2 + \lambda^2}c.$$

Having inserted the numeric values of wavelengths of red and green light, we get the final velocity

$$v = \frac{11}{61}c \doteq 0.18\,c.$$

For comparison, we can also calculate the final velocity using the classic Doppler equation. Let's choose the reference frame of the traffic light. In this reference frame the final observed frequency would be $f = f_0\frac{c+v}{c}$, so the velocity will be

$$v = \left(\frac{f}{f_0} - 1\right)c = \left(\frac{\lambda_0}{\lambda} - 1\right)c = 0.2\,c.$$

35 In the beginning, the system is in equilibrium. There is the same mass on the both sides. Katy's mass is m, Mike is two time heavier, so his mass is $2m$. We added to Katy's side a weight of the same mass. Therefore, mass of $2m$ is on both ends of the rope.

The question is, what happens when the lighter frog starts climbing up with the velocity v in the rope's reference frame? The solution lies in answering the question how exactly the lighter frog has accelerated up to the velocity v? Accelerating frog causes the force $F(t)$ to start acting on one side of the pulley. This force can have arbitrary development.

According to the 3rd Newton's law a force of magnitude $F(t)$ acts on the Katy in the direction upside. The pulley is massless and frictionless. Therefore, net force and net torque acting on the pulley is always zero. For that reason, the force of magnitude $F(t)$ acts on the heavier frog as well. Both, lighter and heavier frog receives impulse J.

$$J = \int_{t_1}^{t_2} F\,\mathrm{d}t,$$

Impulse J causes the change of the momentum $\Delta p = J$ on the both sides of the pulley.

Denote the velocity of the heavier frog as u and it is directed upward. Also, it is the velocity of the rope. From the fact that the momentum has to be the same on both sides of the pulley, we get

$$2mu = m\,(v-u) - mu,$$

Thus, $u = \frac{v}{4}$. Therefore, the lighter frog climbs up to the pulley with the velocity equal to $\frac{3}{4}v$ and she is the first who to reach the pulley. It takes her time

$$t = \frac{H}{\frac{3}{4}v}.$$

During this time, the centre of mass of the heavier frog is shifted by distance $h = ut = \frac{H}{3}$. Therefore, the distance between the frogs is $\frac{2}{3}H$.

Let us note a short comment to the momentum of the system and another type of the solution (using the principle of conservation on angular momentum). Notice that center of mass of the system (two frogs and the rope) is moving upwards. It means that the momentum of the system has changed. Therefore, it cannot be constant. Why is not the total momentum of a system constant (conserved)? When the monkey accelerates and thereby acts with the force on the rope, it causes a reaction force in the pulley. The reaction force in the pulley is bigger i. e. the external force is acting on the system. However, the momentum of the system will not change anymore after the frog starts moving with the constant velocity.

We can solve this problem using the principle of conservation of angular momentum. There is no external torque acting on the system. Therefore the net angular momentum of the system is conserved. If we set the net angular momentum equal to the zero for a whole duration of the motion (frog climbing), we get

$$2mu = m(v - u) - mu.$$

It is exactly the equation for the momentum on both sides of the pulley – just look a little bit above!

36 Have you ever heard about solar sail? The same principle is applied here – light reflected off a surface results in a force. Since light produces force, it has to posses momentum. Thanks to some very clever people in the past, we can today view light as some kind of a stream of particles, photons, every single of which has an energy

$$E = hf$$

and momentum

$$p = \frac{E}{c}.$$

The photon stream is reflected off the inner side of the wheel at an angle of $90° - \alpha$ to a normal to the surface. Each reflection causes a change in the photon's momentum

$$\Delta p = 2p\cos(90° - \alpha) = 2\frac{E}{c}\sin\alpha.$$

The flight of the photon between two reflections takes time

$$\Delta t = \frac{2R\sin\alpha}{c}.$$

Now, the crucial idea is applied. Since the photons are discrete particles, the force on the wheel is irregular. But as the photons travel so fast, they are hitting the wheel very often. And since the time Δt between two collisions is really short, we can calculate the average force, neglecting the fluctuations, and considering the force to be constant, equal to this average force.

We can express the average force caused by a single photon as

$$\frac{\Delta p}{\Delta t} = \frac{2\frac{E}{c}\sin\alpha}{\frac{2R\sin\alpha}{c}} = \frac{E}{R}.$$

And because the photon is wildly flying around the whole wheel, we can use the same argument as before and consider the force uniformly distributed across the whole wheel. We can also note that the force is independent of the angle by which the laser shines – smaller angle means collisions happen more often, but each requires smaller change of momentum.

When Saymon shines his laser for a time t, he creates a beam with an energy Pt. This beam will consist of $\frac{Pt}{E}$ photons, each with the same energy E. Since the same arguments can be used for all photons, we can find that the force caused by $\frac{Pt}{E}$ photons is $\frac{Pt}{E}$ times greater, thus we can express the total force as

$$F = \frac{Pt}{E}\frac{E}{R} = \frac{Pt}{R}.$$

We can again note that the force does not depend on the energy of a single photon, since greater energy means greater force, but fewer photons in the whole beam. This is of course the force acting in the radial direction, so we must determine the tensile force. To do this, we can imagine that the wheel expands slightly, and look at the work done by all the present forces. The law of conservation of energy will then allow us to find the unknown force T[3]. If we increase the wheel's radius by Δr, its circumference will increase by $2\pi\Delta r$. The radial force will do the work

$$W = \frac{Pt}{R}\Delta r$$

and the tensile force

$$W = T 2\pi \Delta r.$$

Their equality provides us with the unknown force

$$T = \frac{Pt}{2\pi R}.$$

37 If we want to determine the moment of inertia of Samuel's dried up onion, we have to remember several facts. The first one is trivial, moment of inertia of solid sphere is $\frac{2}{5}mR^2$, where m is mass of a solid sphere. The second one is, that moment of inertia is aditive – if the final object is composed of multiple parts, resultant moment of inertia about arbitrary axis is sum of moments of inertia of individual parts about the same axis. The last thing to remember is also easy. If every dimension is increased k-times, mass will increase k^3-times. In other words, you have to know scaling.

After this quick intro, we can finally begin calculation. Moment of inertia of a solid sphere can be composed of moment of inertia of the onion and its dried up part, which has shape of onion, but two-thirds dimensions. Let's denote moment of inertia of the onion I. General formula for moment of inertia is $c \times MR^2$, where c is constant determined by shape and chosen axis, hence moment of inertia of dried up part is $\left(\frac{2}{3}\right)^5 I$. Therefore, moment of inertia of solid sphere is

$$\frac{2}{5}mR^2 = I + \left(\frac{2}{3}\right)^5 I.$$

In this case, m is mass of solid sphere. If the mass m is determined by scaling of onion's mass M, equation $m = M + \left(\frac{2}{3}\right)^3 M$ can be written. Trivial calculation gives

$$I = \frac{2}{5}\frac{1+\left(\frac{2}{3}\right)^3}{1+\left(\frac{2}{3}\right)^5}MR^2 = \frac{126}{275}MR^2.$$

[3] principle of virtual work

38 Contrary to the identical circuit connected to DC-generator, the result is significantly influenced by capacitors and coils. In AC circuits, Ohm's law is applied to peak voltage U and peak current I

$$U = |Z|I,$$

where $|Z|$ is magnitude of impedance. Therefore, it is sufficient to calculate impedance of the circuit.

It is easy to notice that two central branches consisting of coils and resistors are symmetric. Therefore, potentials at both ends of the central capacitor have to be identical, hence voltage and current flowing through the capacitor are both zero. In other words, central branch with capacitor can be eliminated. Our problem is now reduced to calculation of impedance of series and parallel circuits. Total impedance of AC circuit is

$$\frac{1}{Z} = \frac{1}{R} + \frac{2}{R + i\omega L} + i\omega C = \frac{3R - \omega^2 RLC + i\left(\omega R^2 C + \omega L\right)}{R^2 + i\omega RL}.$$

Magnitude of impedance is

$$|Z| = \sqrt{\frac{R^4 + \omega^2 R^2 L^2}{\left(3R - \omega^2 RLC\right)^2 + \left(\omega R^2 C + \omega L\right)^2}},$$

hence

$$I = U\sqrt{\frac{\left(3R - \omega^2 RLC\right)^2 + \left(\omega R^2 C + \omega L\right)^2}{R^4 + \omega^2 R^2 L^2}}.$$

39 We start by answering the question how Jonas's wheel was created. Let's suppose that it is homogenous with the area density σ. The original (full and unbroken wheel) has a mass $\sigma\pi r^2$. The centre of mass is in the middle and its coordinates are $[0; 0]$. The hole inside the wheel would have the mass $\sigma\pi\left(\frac{r}{2}\right)^2$ and its centre of mass is in the point $\left[0; \frac{r}{2}\right]$. That means that the broken wheel has the mass $\frac{3}{4}\sigma\pi r^2$. The x coordinate of the centre of mass is zero. The y coordinate can be calculated from equation for torques

$$y = \frac{\sigma\pi r^2 \cdot 0 - \sigma\pi\frac{r^2}{4} \cdot \frac{r}{2}}{\frac{3}{4}\sigma\pi r^2} = -\frac{r}{6}.$$

The minus sign represents that the y coordinate is on the opposite side to the hole in the wheel. We denote the distance between the centre of mass of the broken wheel and the centre of mass of the unbroken wheel as t, $t = -y = \frac{r}{6}$.

Remember, that we focus on small oscillations. What happens when the wheel is deflected by an angle φ? Or in other words, what happens if the wheel's contact point with the pit moves by an arc with lenght $r\varphi$. The angle between the new and the old contact point is $\alpha = \frac{r}{5r}\varphi = \frac{\varphi}{5}$ with respect to the centre of the pit.

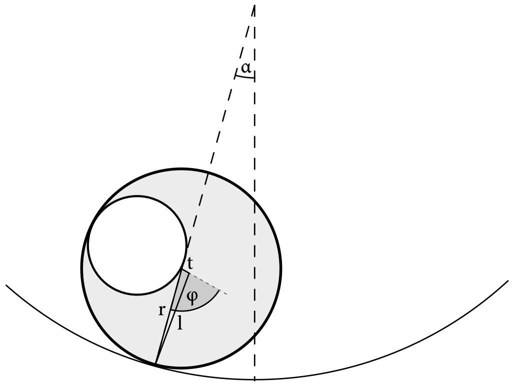


Figure 6: *Geometry of Jonas' oscilating wheel*

Potential energy with respect to the bottom of the pit is

$$E_p = mgh = \frac{3}{4}\sigma\pi r^2 gh,$$

where the height h is

$$h = 5r\left(1-\cos\alpha\right) + r\cos\alpha - t\cos(\varphi-\alpha) = r\left[5\left(1-\cos\frac{\varphi}{5}\right) + \cos\frac{\varphi}{5} - \frac{1}{6}\cos\frac{4\varphi}{5}\right].$$

In this case, the kinetic energy can be expressed as the rotational energy around the axis of rotation. However, the axis of rotation is constantly changing. Therefore,

$$E_k = \frac{1}{2}I_A\omega^2,$$

where I_A is the moment of inertia around the actual axis. The distance of the centre of mass from this axis is l. From the law of cosines we get $l^2 = r^2 + t^2 - 2rt\cos\varphi = r^2\left(1 + \frac{1}{36} - \frac{1}{3}\cos\varphi\right)$.

Only one unknown remains - the moment of inertia of the Jonas's (broken) wheel. Firstly, we have to calculate the moment of inertia around the centre of mass, I_T. Secondly, we will use the Parallel axis theorem (Steiner's theorem) and the fact that we can sum up the moment of inertia of the broken wheel and the one which fell out, in order to get the moment of inertia of the full, undamaged wheel. The moment of inertia of a disk is $\frac{1}{2}mR^2$, so

$$\frac{1}{2}\sigma\pi r^2 r^2 = \left(I_T + \frac{3}{4}\sigma\pi r^2 t^2\right) + \left(\frac{1}{2}\frac{1}{4}\sigma\pi r^2\left(\frac{r}{2}\right)^2 + \frac{1}{4}\sigma\pi r^2\left(\frac{r}{2}\right)^2\right).$$

We get that the moment of inertia around the centre of mas is $I_T = \frac{37}{96}\sigma\pi r^4$. Next, we use Steiner's theorem again. Subsequently, for the moment of inertia around the current axis of the rotation, we get

$$I_A = I_T + \frac{3}{4}\sigma\pi r^2 l^2 = \sigma\pi r^4\left(\frac{37}{32} - \frac{1}{4}\cos\varphi\right).$$

The small oscillations are of our concern. Therefore, we estimate I_A. We will use approximation $\cos\varphi \approx 1$[4]. Afterwards, our equation is

$$I_A = \frac{29}{32}\sigma\pi r^4.$$

In the end, we evaluate the net energy of the broken wheel[5]. For the potentional energy, we use the approximation of the term $\cos x$ by the Taylor's polynom up to the second order of precission ($\cos x \approx 1 - \frac{x^2}{2}$). We get

$$E = \frac{1}{10}\pi\sigma r^3 g\varphi^2 + \frac{29}{64}\pi\sigma r^4\omega^2 = \text{const.},$$

what is the energy of harmonic oscillator with the period which is defined by the constants in front of φ^2 and ω^2.

Finally, we find out that Jonas's broken wheel oscilates with the period

$$T = 2\pi\sqrt{\frac{\frac{29}{64}\pi\sigma r^4}{\frac{1}{10}\pi\sigma r^3 g}} = 2\pi\sqrt{\frac{145}{32}\frac{r}{g}}.$$

40 Let us denote the distance between the counterweight and the pivot as x. The problem doesn't require us to maximize the acceleration for any particular angle of the rod's tilt, which is alright, since the acceleration only depends on the angle by a factor of $\cos\alpha$. That means that maximizing the acceleration for any particular angle maximizes it for all angles. Let us write down the expression for the linear acceleration of the projectile at zero angle of tilt

$$\begin{aligned} a &= (L-x)\,\varepsilon \\ &= (L-x)\,\frac{\tau}{I} \\ &= (L-x)\,\frac{Mgx - mg\,(L-x)}{Mx^2 + m\,(L-x)^2}. \end{aligned}$$

Our task is to maximize this expression with respect to the variable x. We will first rearrange it into a more convenient form

$$a = \frac{MLx}{Mx^2 + m\,(L-x)^2} - 1.$$

If we want to maximize this expression, we can disregard the constant term. Furthermore, we can simplify our work by using the fact that a maximum of a positive expression implies a minimum of its reciprocal. Hence we are to look for a minimum of the expression

$$a = \frac{Mx^2 + m\,(L-x)^2}{MLx}.$$

We can now proceed in two different ways – either by using calculus and differentiating, or using so-called *inequality of arithmetic and geometric means*. The later states that an arithmetic mean of any sequence of

[4] from the Taylor's polynom
[5] except the constant element

numbers is always greater than or equal to their geometric mean. For the case of two numbers

$$\frac{a+b}{2} \geq \sqrt{ab},$$

what can be proven by simple squaring of both sides. If we substitute $a \to ax$ and $b \to \frac{b}{x}$ into this, the right-hand side does not change and the inequality will acquire the form

$$ax + \frac{b}{x} \geq 2\sqrt{ab}.$$

We are lucky and our original expression can be (almost) rearranged, so that it matches the left side of the inequality

$$a = \frac{m+M}{ML}x + \frac{mL}{Mx} - \frac{2m}{M}.$$

Fortunately, the leftover term is only a constant, so it can be ignored. From the AM-GM inequality, it follows that for any number x, this expression is always greater than or equal to

$$2\frac{\sqrt{(m+M)m}}{M}.$$

But we want to find such x, that the expression is as small as possible which implies equality. Thus we obtain an equation for x

$$\frac{m+M}{ML}x + \frac{mL}{Mx} = 2\frac{\sqrt{m\,(m+M)}}{M},$$

which can be rearranged into a quadratic equation whose solution is

$$x = \sqrt{\frac{m}{m+M}}L.$$

For completeness, we will also demonstrate the solution by using derivatives. An expression attains its minimum when its derivative is zero

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{Mx^2 + m\,(L-x)^2}{MLx} = \frac{2Mx - 2m\,(L-x)}{MLx} - \frac{Mx^2 + m\,(L-x)^2}{MLx^2} \stackrel{!}{=} 0$$

Solving this equation for x leads us to the same result as above.

Answers

1 $t = \frac{\sqrt{(a+b)^2 + c^2}}{v}$

2 10.39 m/s

3 Martin, o 6 minút

4 34.5 %

5 $\frac{t_1 - t_2}{2}$

6 $\frac{2\pi r^2 v^2 \rho}{g}$

7 11 cm

8 $\frac{u}{u+v} = \frac{1}{1+\frac{v}{u}}$

9 $\left[\frac{2521}{177}a; 6.5a\right]$

10 80 cm

11 4.5 m

12 60°

13 $3\frac{U}{R}$

14 $\frac{7}{9}$

15 0°

16 95 mm

17 2.17 g

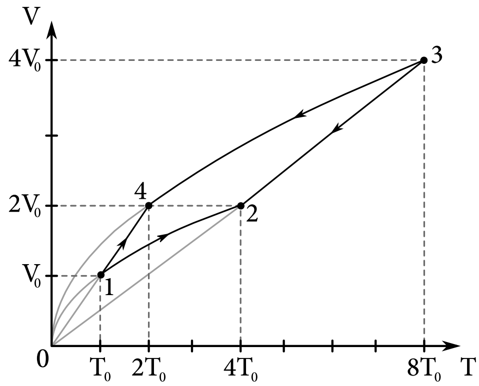
Uznajte výsledky, ktoré sa líšia od uvedenej hodnoty o najviac 0.01 g*.*

18 $\frac{5M}{4m}$

19 $\frac{4}{5}$

20 230.9 km. *Accept values between* 229.9 km *and* 231.9 km.

21 *Dbajte na to, aby odovzdané grafy mali vyznačené všetky dôležité hodnoty na osiach, šípky popisujúce priebeh deja mali správny smer a lineáry aj odmocniny po predĺžení prechádzali cez počiatok. Pokojne sa pýtajte riešiteľov na tvar kriviek, tzn. či naozaj vedia, že sú to odmocniny.*



22 116 m^2

23 $\frac{\pi R^3 \rho}{3}$

24 $\arcsin\left(\frac{m}{M+m}\frac{\mu}{\sqrt{1+\mu^2}}\right)$

25 $\frac{4\varepsilon}{(1+\varepsilon)^2}$

26 29.43 m/s.

27 1 – the magnitude of the field is the same

28 1.6 dm^3

29 $\frac{\sqrt{2}-1}{\sqrt{2}+1} = 3 - 2\sqrt{2} = \left(\sqrt{2}-1\right)^2$

30 0, the field is zero.

31 $\sin\varphi_0$

32 3 538 765 km^2. *Accept values in the interval* (3 528 765 km^2; 3 548 765 km^2).

33 $T = \frac{R^{3/2}}{\sqrt{GM}}\left(\pi + \sqrt{2}\right)$

34 $\frac{11}{61} \doteq 0.18\,c$

35 The lighter frog gets to the pully first and her distance from the heavier one will be $\frac{2}{3}H$.

36 $\frac{Pt}{2\pi R}$

37 $\frac{126}{275}MR^2$

38 $U\sqrt{\frac{\left(3R-\omega^2 RLC\right)^2+\left(\omega R^2 C+\omega L\right)^2}{R^4+\omega^2 R^2 L^2}} = U\sqrt{\frac{\left(3-\omega^2 LC\right)^2+\left(\omega RC+\omega \frac{L}{R}\right)^2}{R^2+\omega^2 L^2}} = U\frac{\sqrt{(3R^2+\omega^2 L^2)^2+(\omega R^3 C-2\omega RL+\omega^3 RL^2 C)^2}}{R^3+\omega^2 RL^2}$

39 $2\pi\sqrt{\frac{145}{32}\frac{r}{g}}$

40 $x = \sqrt{\frac{m}{m+M}}L$