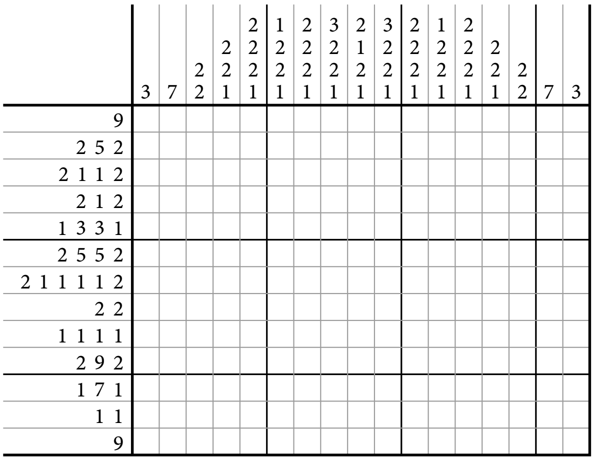
Problems

1 Granny Justine was reading the Geochimica et Cosmochimica Acta journal and found a nonogram in there. After she solved it she felt a sudden urge to find the centre of mass of the picture. How far is it from the geometric centre of the picture if all the coloured squares have equal mass and the empty squares are massless if the side of each small square is 1?



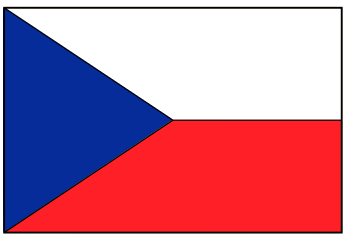
2 Marek joined a live video stream thirty minutes after it began. He set the playback speed to 1.75×. How long will it take him to catch up with the live stream?

Submit the answer in format `HH:MM:SS`.

3 A hiker has just reached the summit of a high peak. The total altitude difference of his trail was 1302 metres and his total ascent was 1447 metres. How many metres will he have to ascend on his way back if he takes the same route?

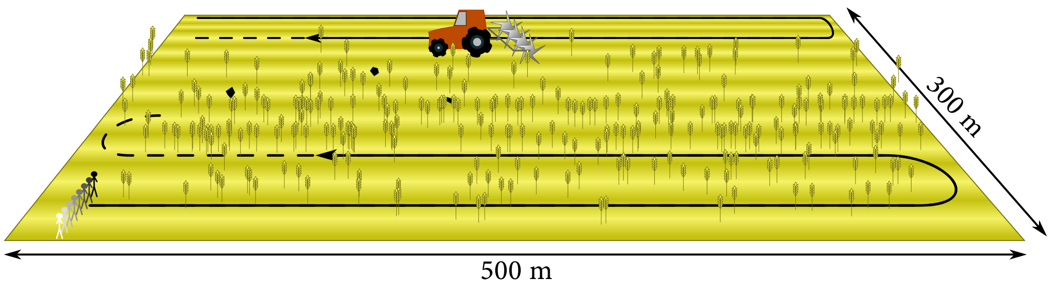
4 In honor of a recent holiday that we celebrated, we assembled the Czechoslovak flag with 54 cm wide and 36 cm tall from three coloured pieces of sheet metal. The blue triangle extended to the half of the flag's width. When on display outside, the flag was illuminated by the Sun and each of its parts reached a different temperature: the white part 30 °C, the red one 70 °C and the blue one was heated to 90 °C.

What will the temperateur of the flag be if we isolate it from its surroundings and wait until the temperatures are equalized?



5 A team of eight people is searching a field for a recently fallen meteorite. They are walking in a skirmish line. Each person scans a strip 2.5 m wide while walking at a speed of 1 m/s. Every time they arrive at the end of the field they quickly turn around, move to an adjacent strip of land and continue the search there. The field is 500 m long and 300 m wide.

However, at the same time as the team started their search, a tractor started ploughing the field at a distance of 300 m away. Its plough is 10 m wide and it is moving at 18 km/h. Upon reaching the end of the field it turns around very quickly as well. What fraction of the field will remain **unscanned** if the search has to stop once **any** member of the team meets the tractor?



6 Once again, comfort prevailed over ecology… Marcel, Kevin, Patrick and Jacob are driving to the train station, each in his own car. When they are driving through the village at the maximum allowed speed 50 km/h they occupy a section of the road 150 m long. If everyone drives exactly the same, how long is the section they will occupy outside the village after they accelerate to a velocity of 90 km/h?

Neglect the lengths of the cars.

7 Daniel chose two random resistors from his tinkering box and plugged them into an electrical circuit, first in series, then in parallel. In the evening the cleaning lady wandered into his lair and on the table, among other things, she found a small piece of paper with values 4 Ω and 25 Ω scribbled on it.

What were the resistances of the resistors?

8 At a medieval castle a tourist guide shows the famous well to curious visitors. The guide, underestimating the mathematical capabilities of the visitors, presents a *simplified* calculation of its depth: the visitor should throw a coin or a stone into the well and count the time until they hear the sound of the coin hitting the bottom. Then, to find the depth, they should multiply this time by some constant k. Of course, the guide set the value of the constant k so that this actually works for the particular well. Express how would you find the depth of the well from the constant k and the acceleration due to gravity g.

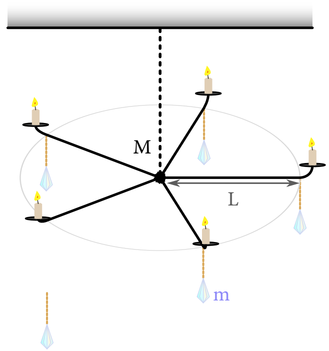
In all medieval wells the speed of sound is infinite.

9 Justine prepared a solution of acetone in water. However, she needed to leave the lab for a moment. When she came back, she found out that one third of the acetone mass and one tenth of the water mass had evaporated. Justine measured that the mass fraction of acetone in the solution decreased by one sixth.

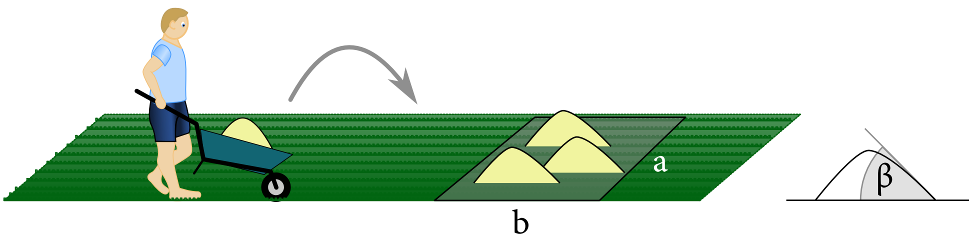
What was the original mass fraction of acetone in the solution?

10 Vicky has a chandelier consisting of a point mass M from which five very light rods with length L extend in one plane, separated by equal angles. At the end of each rod there is a candle with mass m. The entire chandelier is suspended from the ceiling by a long chain that is connected to the central point mass.

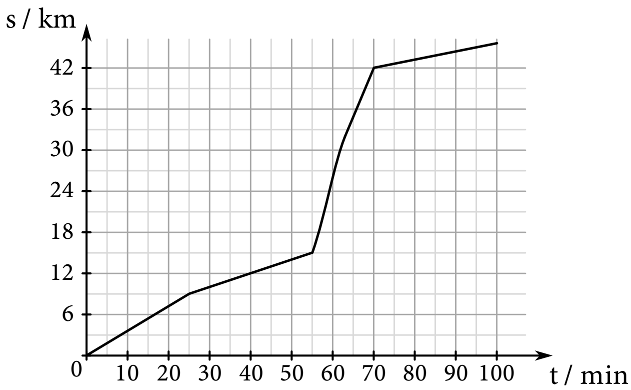
Suddenly one of the candles falls off. What will be the new angle between the plane of the rods and the horizontal plane after the system reaches equilibrium?



11 William bought a rectangular parcel with sides a and b, where $a \geq b$. How much sand can he pour there so that all of it remains on his premises? The angle of repose of sand is β.

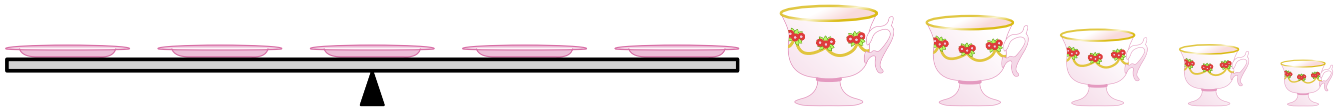


12 Jacob was trying out a new app for public transports enthusiasts. Among other things it records the travelled distance and displays the average velocity of the phone measured in km/h since the beginning of the trip. Jacob spent last night riding the buses and today he plotted a graph showing the travelled distance plotted against time, as shown below. What was the highest average velocity the app was showing during Jacob's trip?

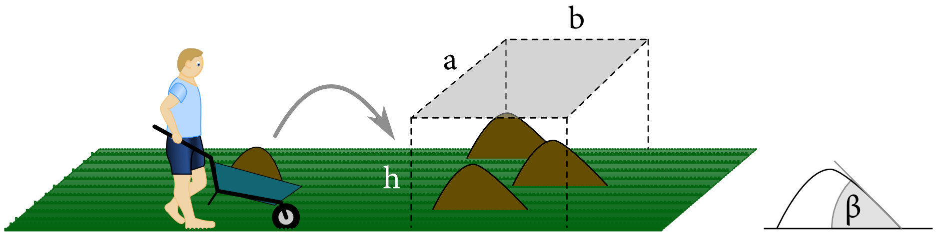


13 Theresa bought a special tea set in England. The tea set consists of a seesaw balanced in its centre and five saucers placed at equal distances. Now Theresa invited her friends to a tea party. Five friends sit in alphabetical order at her special tea set. She brings five tea cups with mass ratios 1 : 2 : 3 : 4 : 5 and places them on the saucers.

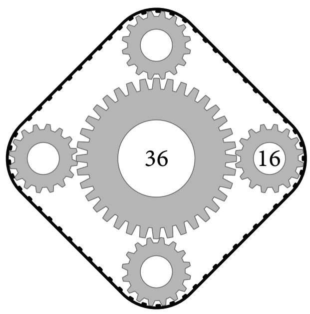
In how many distinct ways can Theresa arrange the tea cups for her friends if the seesaw must remain balanced?



14 William also wants to build a monument on his parcel. The base of the monument will be a rectangular concrete plane with sides $a = 10$ m and $b = 7$ m. Monument has to be placed in height $h = 3$ m above the current surface and must be supported under its entire area. What is the minimal volume of soil William will need to place under the concrete base, if the angle of repose of soil is $\beta = 45°$?

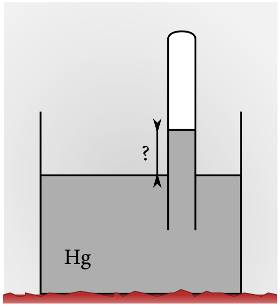


15 On his birthday Andrew got a construction set. He instantly built a special machine: he took a wheel with 36 cogs and placed four smaller ones, with 16 cogs, around it. Then he put a belt around the whole construction. How many times does he have to turn the large cogwheel in order to make the belt go around once? The axes of all the cogwheels are fixed.

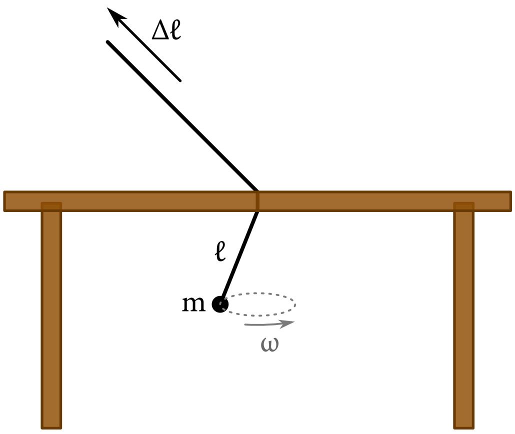


16 Astronaut Thomas has gone on a trip to Mars and has brought his hereditary barometer with a mercury column with him. The column was 760 mm high on the Earth. How high is it on Mars where the atmospheric pressure is 600 Pa?

Neglect the rotation of both planets.

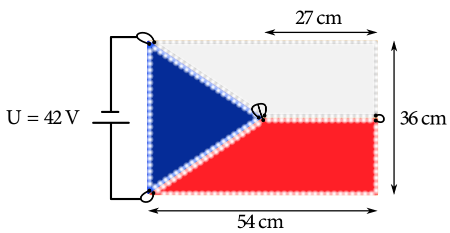


17 Wilhelmina has stuck a string though a hole in her table so that the length of a string below the table is ℓ. Then she hung a weight m on the string and made it rotate on a circular path with angular frequency ω. Understandably, she has to hold the upper end of the string in her hand. What amount of work will she do if she pulls the string by a small distance $\Delta\ell$ towards her?



18 A metal flag is nice and can provide much comfort, but you can't see it when it's dark. That's why we took the Czechoslovak flag with width 54 cm and height 36 cm and illuminated it by luminous strips with electric resistance 30 Ω/m so that each color was enclosed by a strip from all sides and the strips were connected in the vertices of the flag. Then we connected the two leftmost corners of the flag to an input voltage 42 V.

Our security technician, however, was not convinced by the safety of this circuit, so he turned off the voltage source, randomly chose a point on the strip and inserted a fuse with nominal current rating of 2 A. What is the probability that the fuse will blow, if the input voltage is turned on again?

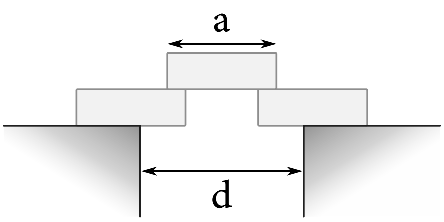


19 Lucy the engine driver activated the emergency brake on her train. The brakes are operated pneumatically and have a slight delay: the wheels of the locomotive are blocked immediately, those on the first car a second later and those on every consecutive car 1 s after the previous one.

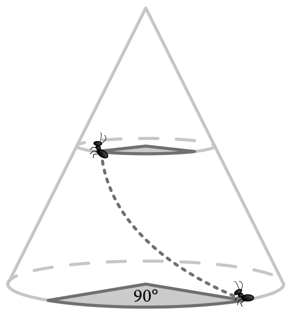
The train has five cars, each weighing 20 t, the locomotive weighs 100 t and the coefficient of dynamic friction between the wheels and the rails is 0.2. What is the total distance travelled by the braking train from an initial speed of 72 km/h?

20 Adam likes building bridges. This time he took three identical blocks with edges $a \geq b \geq c$. He wants to know the length of the longest possible gap over which he can build a bridge with only these three blocks. Friction between all blocks and between the blocks and the floor is small and Adam can not rely on it.

The front faces of the blocks must lie in one plane.

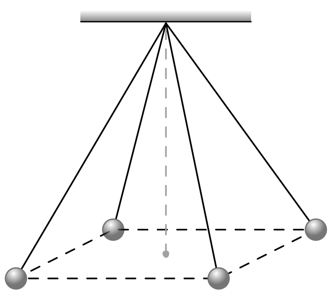


21 An ant is climbing up a cone-shaped sugar loaf with radius 1 m and height 2 m. Currently he is standing at the base of the loaf exactly to the southeast from the top. What is the shortest distance he has to climb if he wants to get to the southwestern side of the loaf to the altitude of 1 m?



22 Paula found at home four identical homogeneous sticks with mass 2 kg and length 2 m. She hung them from one point on the ceiling. Then she attached one massless point charged to 10^{-4} C to the end of each of the sticks so that they would repel each other. Find the angle between the sticks and the plumb line in the state of equllibrium.

This problem does not have a closed-form solution. Feel free to use calculator. Submit you answer in degrees with precision of at least one decimal place.

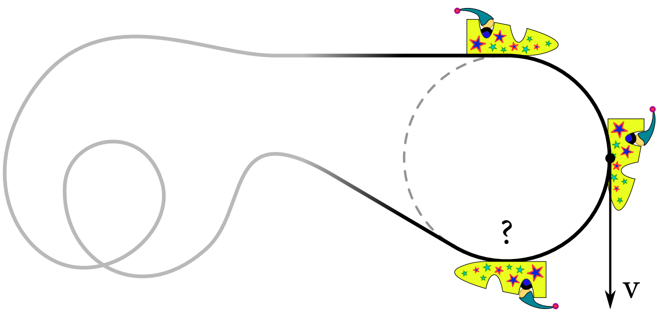


23 A lorry is driving down the road at the speed of 50 km/h. A fly is flying in the opposite direction at the same speed. The collision ends up exactly as one would expect and the fly is turned into goo. By how much does the goo's temperature increase if the specific heat capacity of the fly is 3 kJ/(kg·K)?

Assume that the windshield is perfectly rigid and does not warm up.

24 Matthew has successfully finished his studies so now he can finally work in an amusement park. His favourite attraction is a rollercoaster with a vertical half-loop. Matthew sits down in a car at the top of the loop and slightly pushes himself off the railing to move forward. When he is halfway down, his speed is v. What g-force will Matthew feel when he will be in the lowermost point of the half-loop?

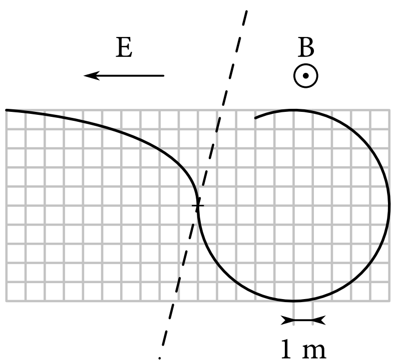
Neglect the air resistance, friction between the car and the loop and all other nasty forces.



25 In Joe's room several springs with unknown rest length are hanging from the ceiling. The springs are close to each other. When Joe hangs on one spring, he remains 50 cm above the floor. When he hangs on two springs, he remains 140 cm above the floor.

How high will Joe be hanging if he grabs onto three springs? The ceiling in Joe's room is 250 cm high.

26 Patrik found a strange electrically charged ball. To learn more about it, he had it analysed in a modern electromagnetic analytic chamber. The chamber is well shielded from gravity and is divided into two parts: in the left half there is a homogeneous electric field with intensity $E = 0.8\ \mathrm{V/m}$, in the right half is a homogeneous magnetic field with intensity $B = 0.4\ \mathrm{T}$. The strange ball was insected into the chamber and its trajectory is shown in the picture. What was the initial speed of the ball?



27 On the shore of Nile with slope α, at a height h above the water level, two homogeneous spherical hippos are sunbathing. Each hippo has a spherical stomach located in the centre of its body. One of the hippos is hungry and the other one is already full of a mix of grass, water and lost tourists, whose density is the same as that of the hippo itself. The mass of a hungry hippo is one eighth less than the mass of a full one.

When they have had enough sunbathing, they start rolling down into the water. Which one will be faster? And how many times?

The contents of a hippo's stomach rotate with the rest of the hippo.

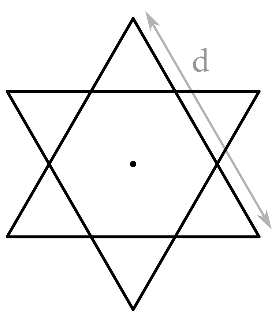
28 Lucy inflated a beach ball of mass 100 g and radius 10 cm with air at atmospheric pressure. Then she started slowly submerging it into water. What is the greatest possible depth into wich it can be submerged so that it will still emerge on its own, if the temperature of water is constant everywhere?

29 We have cut out a circular disc with radius r and a square with sides a out of a thin metal sheet with area density σ. Then we have poked a rod through the centres of both of our objects. We have made the disc rotate with angular frequency ω and then brought it in contact with the square. What will be the final angular frequency of both objects after the friction force causes them to rotate with the same angular frequency?

30 Daniel found that flowing electric current heats up a resistor. His favourite resistor connected to a voltage-fixed power supply is also heated up – after a long time, its temperature stabilized at T_1. After this experiment, he did the exact same experiment with another resistor. This one was made from the same material as his favourite one, but all the dimension were twice as big. What is the stable temperature for the second resistor?

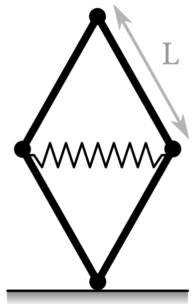
In both cases, there was a fan blowing onto the the resistor; that blowed off the heated air and held the temperature of the air surrounding the resistor T_0. Neglect the radiation heat transfer.

31 During an evening walk through the streets of Bratislava, Sabinka noticed that the Christmas decorations were already up. At first, she wondered why does Christmas begin in November again, but then she spotted an interesting decoration shaped as a six-pointed star. It was made out of six rods with length d and mass m. As a true physicist, she immediately calculated its moment of inertia with respect to the axis going through the centre and perpendicular to the plane of the star. What answer did she get?

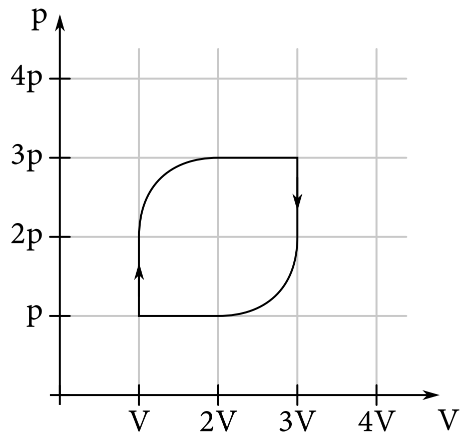


32 In addition to a promising career as a physicist, Dave took up modern art. He ran to the attic and carried down four sticks with a mass of m and a length of L. and one spring with zero rest length. Then he joined the sticks into a quadrilateral so that they could rotate freely in the joint, while always remaining flat.

Ge connected the two opposite corner points with a spring and placed his work of art on one corner point so that the spring was horizontal. What is the stiffness of the spring if it stretches to length L?

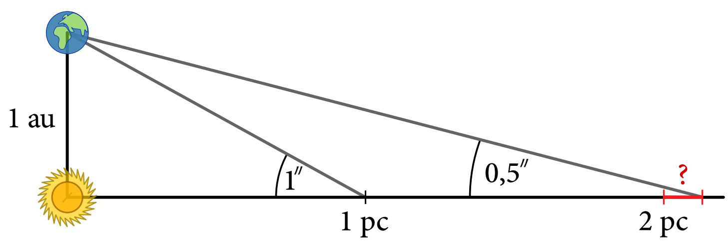


33 In the midst of the energy crisis, Jaro pulled out his great-great-grandfather's steam engine from the barn. All it took was a few quick fixes and it worked like new. The machine now operates according to the cycle shown in the pV diagram. What is its efficiency if the gas can be treated as an ideal monoatomic gas?



34 One parsec was defined as the distance, from which the angular radius of the Earth's orbit is one arc-second ($1''$). Martin is trying to calculate the distance to a star that is 2 pc away, as if the angular radius of the Earth's orbit when viewed from there were $0.5''$. Martin realizes that this is not entirely correct, but since these angles are extremely small, he knows that the relative error will be negligible. Still, he would like to know: how much does this distance differ from an actual length of 2 pc?

If your calculator does not cooperate, you may want to try some Taylor series.



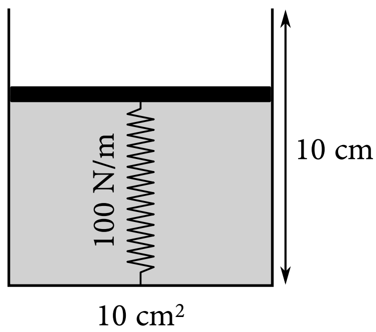
35 A perfectly black spherical planet is being heated up by its own power P. If we leave it on its own, its temperature will stabilize at T. Now we cover the planet with a thin reflective spherical layer of the same radius as the planet. The layer reflects 80 % of incident radiation back to the planet and absorbs the rest. What is the equilibrium temperature of the layer, if it's not directly touching the surface of the planet?

36 Thomas is planning to construct a huge telescope. As he was slightly worried about breaking the expensive primary mirror, he decided to use liquid mercury instead of glass: he poured the mercury into a large flat bowl and spun it up to angular speed ω.

What is its focal distance when placed in a homogeneous gravitational field with magnitude g?

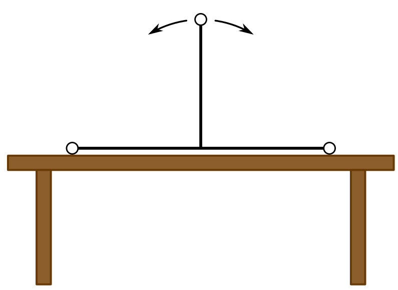
37 Marcel has enclosed a spring with stiffness 100 N/m and zero rest length in a cylinder with a base of area 10 cm^2 and height 10 m. The cylinder is filled with an ideal diatomic gas. Marcel attached one end of the spring to the bottom of the cylinder and the other end to a massless piston which hermetically enclosed the cylinder. When he was closing the cylinder with the piston, there was standard atmospheric pressure in the cylinder, which understandably increased when the piston reached its equilibrium height and temperature.

What is the apparent stiffness of the enclosed spring for small displacements from equilibrium?



38 Paula is assembling mechanical systems again. This time, she has three homogeneous sticks with mass 2 km, length 2 m and each of them has a massless point charge of 10^{-4} C on one of its ends. She has glued two sticks onto a horizontal table so that they were parallel and touched each other with their ends that do not have a charge.

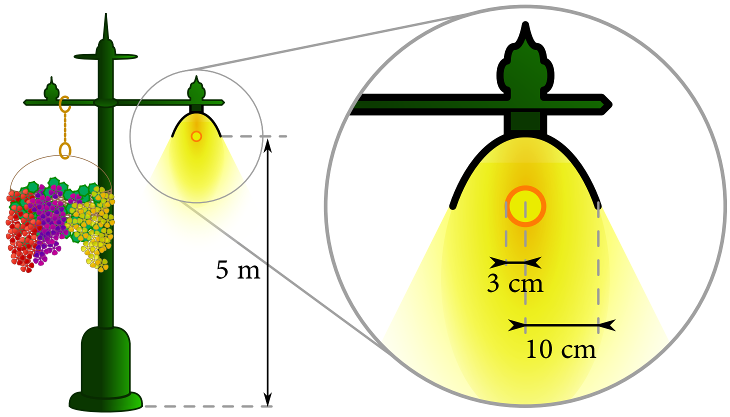
Then she took the third stick and mounted it with a hinge into the point where the previous two sticks touch while the charge of the third stick was on the top end of the stick. It could freely rotate so that all three sticks remained in the vertical plane. Then she slightly displaced the third stick from a vertical position towards one of the sticks on the table and observed its small oscillations. What is the period?



39 A solitary streetlamp is standing on the edge of a pier. It has a reflective lampshade in the shape of a paraboloid, the radius of its opening is 10 cm, its depth is 7 cm and its inner surface is perfectly reflective.

In its focus there is a spherical light bulb with radius of 3 cm, that isotropically emits a total flux of 10 klm. What is the illuminance of the point on the road directly under the light bulb?

For small angles you may use the approximation $\sin x \approx \tan x \approx x$. *The light bulb is fully opaque.*



40 Matthew and Jacob are exploring the Moon's surface. Matthew had landed at the North Pole whereas Jacob continued to the Moon's equator. A few moments later, Matthew found a very interesting rock and he wanted to show it to Jacob for analysis. For that he wants to know the smallest speed with which he needs to throw it so that the rock will reach the equator. The mass of the Moon is $M_{☾}$ and its radius is $R_{☾}$.

Solutions

1 To find the centre of mass we need not solve the nonogram at all: we only need to find the distribution of mass, which is completely described by the numbers in the legend. Of course, we *may* solve it, but it costs time, which is at premium during the competition.

We should immediately notice that the nonogram is symmetric around the vertical axis, and that the horizontal coordinate of the centre of mass is the same as that of the centre of the picture. Hence, we only need to calculate the vertical coordinate of the centre of mass using the numbers in the rows. We assign each row its weight $-6, -5, \ldots, 5, 6$ corresponding to its distance from the middle row. Then we sum all the numbers in each row, multiply the sum by the row's weight and then sum it all up.[1] Finally, we divide it by the number of all coloured squares in the nonogram, which conveniently happens to be exactly 100.

Thus we obtain the vertical coordinate of the centre of mass

$$y = \frac{\sum_{j=-6}^{6}\left(j \cdot \sum_i x_{ij}\right)}{\sum_{j=-6}^{6}\left(\sum_i x_{ij}\right)} - 0.5 = \frac{(-6)\cdot 9 + (-5)\cdot(2+5+2) + \ldots + 5\cdot(1+1) + 6\cdot 9}{9 + (2+5+2) + \ldots + (1+1) + 9} = \frac{-17}{100} = -0.17. \tag{1.1}$$

The distance of the centre of mass from the centre of the picture is therefore 0.17 squares.

2 Marek watches the stream at playback speed $v_M = 1.75 v_S$, where v_S is the speed of the streamer's live broadcast. Since the speeds of streamer's and Marek's video do not change, we can use the formula $v_M = f_M/t_M$, where f_M is Marek's current video time and t_M is the time measured from the moment Marek turned on the video. Similarly, we can write $v_S = f_S/t_S$.

Since Marek wants to catch up with the streamer's live broadcast, he has to see exactly what the streamer is broadcasting live. From this we get the condition $f_S = f_M$, therefore

$$v_S t = v_M t_M. \tag{2.1}$$

Furthermore, we know that Marek started at time $t_0 = 30$ min later, that is, $t = t_0 + t_M$. By substituting t_M into the equation 2.1, we get

$$v_S t = v_M (t - t_0). \tag{2.2}$$

Now we substitute $v_M = 1.75 v_S$ and express the time

$$t = \frac{7}{3} t_0 = 70\ \text{min}, \tag{2.3}$$

or, after converting to the hexadecimal system, `01:10:00`.

[1] Real speed devils will notice that the topmost and bottommost rows are the same, and will not lose time with them either.

3 The total altitude difference is the difference between the ascended and descended altitudes. On his way back, the tourist ascends exactly the height he descended before, which is 1447 − 1302 = 145 metres.

4 Apart from a flag, we see three bodies in a thermal contact. They are made from the same sheet metal, so their specific heat capacity c is the same. And mass? Mass is proportional to their surface area. Using simple geometry we can see that if the flag's mass is M, the blue part has mass $\frac{M}{4}$ and each of the two trapezoid parts has mass $\frac{3M}{8}$.

From that, we can construct a calorimetric equation in the stable state. The equation states that if all parts of the flag change their temperature to T, the flag will not receive nor lose any heat:

$$\frac{3M}{8}c(30\,°\mathrm{C} - T) + \frac{3M}{8}c(70\,°\mathrm{C} - T) + \frac{M}{4}c(90\,°\mathrm{C} - T) = 0. \tag{4.1}$$

Now we may simple divide the equation by c and M (naturally, this temperature does not depend on neither mass nor material of the flag) and further simplify:

$$\begin{aligned} \frac{3}{8}(30\,°\mathrm{C} - T) + \frac{3}{8}(70\,°\mathrm{C} - T) + \frac{1}{4}(90\,°\mathrm{C} - T) &= 0\,°\mathrm{C}, \\ \frac{3}{8}\cdot 30\,°\mathrm{C} - \frac{3}{8}T + \frac{3}{8}\cdot 70\,°\mathrm{C} - \frac{3}{8}T + \frac{1}{4}\cdot 90\,°\mathrm{C} - \frac{1}{4}T &= 0\,°\mathrm{C}, \\ T &= 60\,°\mathrm{C}. \end{aligned} \tag{4.2}$$

5 Eight people, each scanning a strip of land 2.5 m wide, can be replaced by a single virtual person scanning a strip of land 20 m wide. The field can be divided into 30 strips, each 10 m wide. We can quickly calculate that it takes $t = 500$ s for the team to go through two 10 m strips and in the meantime the tractor ploughs five strips. After the time t, both the team and the tractor are at the other end of the field to their starting point. After each period t passes, 7 strips have been either searched or ploughed. That means that the team meets the tractor sometime during the $5th$ period.

In time $4t$, they all start from the same side of the field as at the beginning with 28 strips already searched or ploughed. That means that four members of the team would encounter the tractor if they did not stop the search. The team is sad that they have not found anything but at least they can brag that they have searched an area of

$$S_{\mathrm{team}} = 4 \cdot 500\ \mathrm{m} \cdot 2 \cdot 10\ \mathrm{m} = 40\,000\ \mathrm{m}^2. \tag{5.1}$$

Since we are interested in the fraction that could not be searched, we need to express the total area

$$\frac{S_{\mathrm{field}} - S_{\mathrm{team}}}{S_{\mathrm{field}}} = \frac{500\ \mathrm{m}\cdot 300\ \mathrm{m} - 40\,000\ \mathrm{m}^2}{500\ \mathrm{m}\cdot 300\ \mathrm{m}} = \frac{150000 - 40000}{150000} = \frac{11}{15}. \tag{5.2}$$

6 Since all our protagonists drive exactly the same way and they abide by the traffic rules, they all start to accelerate exactly at the point where they exit the village. This means that the distances between the cars will not remain the same. However, what does stay the same are the time intervals between their cars passing through any point on the road. This is true, because the movement of each of the cars looks exactly the same except for a slight time delay. We are only interested in the distance between the first and the last car.

Before the cars started to accelerate, their speed had been $v_0 = 50$ km/h and the distance between the first and the last car had been $d_0 = 150$ m. This means that the time interval between them was $t = \frac{d_0}{v_0}$. After they accelerated to speed $v_1 = 90$ km/h, this interval remained the same, therefore $t = \frac{d_1}{v_1}$. From this we can calculate the new distance between the cars as

$$d_1 = v_1 t = v_1 \frac{d_0}{v_0} = 270 \text{ m}. \tag{6.1}$$

7 First of all we need to determine which resistance corresponds to which circuit. Two resistors connected in series always have a larger resistance than if connected in parallel. When connected in series, the resistance R_S is the sum of individual resistances, therefore

$$R_S = R_1 + R_2 = 25\ \Omega. \tag{7.1}$$

On the other hand, when connected in parallel, the total resistance R_P is

$$R_P = \frac{R_1 R_2}{R_1 + R_2} = 4\ \Omega. \tag{7.2}$$

Now we have two equations of two unknowns. For example, we can express R_1 as $R_1 = 25\ \Omega - R_2$ and plug it into the second equation. This yields a quadratic equation

$$R_2^2 - 25\ \Omega \cdot R_2 + 100\ \Omega^2 = 0\ \Omega^2. \tag{7.3}$$

Both of its solutions, $R_2 = 5\ \Omega$ and $R_2 = 20\ \Omega$, are physically meaningful and can be obtained by merely swapping the two resistors. Thus the answer to the question is the pair of values $5\ \Omega$ and $20\ \Omega$.

8 Even without the help of the (not very gifted in physics) tourist guide, we know something about the depth of the well – that is

$$h = \frac{1}{2} g t^2 \quad \Rightarrow \quad t = \sqrt{\frac{2h}{g}}. \tag{8.1}$$

The tourist guide tells us the relation with his constant k (this one, by the way, is not dimensionless) $h = kt$. With this information, we can express the constant k using the relation for the depth of the well as

$$k = \frac{1}{2} g t = \frac{1}{2} g \cdot \sqrt{\frac{2h}{g}} = \sqrt{\frac{hg}{2}}. \tag{8.2}$$

From this we can express h using only k and g as

$$h = \frac{2k^2}{g}. \tag{8.3}$$

9 At the beginning, there was a mass m_{A0} of acetone and m_{W0} of water in the solution. The mass fraction can be obtained as

$$X = \frac{m_{A0}}{m_{A0} + m_{W0}}. \tag{9.1}$$

After Justine returned, there was only m_{A1} of acetone. One third of the acetone and one tenth of water had evaporated, so

$$m_{A1} = \frac{2}{3} m_{A0} \qquad \text{and} \qquad m_{V1} = \frac{9}{10} m_{V0}. \tag{9.2}$$

The new mass fraction is

$$\frac{5}{6} X = \frac{m_{A1}}{m_{A1} + m_{W1}}. \tag{9.3}$$

From this equation, we express X and plug it into the equation 9.1. Then we substitute masses from equation 9.2 to obtain

$$\begin{aligned} \frac{m_{A0}}{m_{A0} + m_{W0}} &= \frac{6}{5} \frac{m_{A1}}{m_{A1} + m_{W1}} \\ \frac{\frac{3}{2} m_{A1}}{\frac{3}{2} m_{A1} + \frac{10}{9} m_{W1}} &= \frac{6}{5} \frac{m_{A1}}{m_{A1} + m_{W1}} \\ \frac{3}{2} {m_{A1}}^2 + \frac{3}{2} m_{A1} m_{W1} &= \frac{6}{5} \frac{3}{2} {m_{A1}}^2 + \frac{6}{5} \frac{10}{9} m_{A1} m_{W1} \\ \frac{3}{2} m_{A1} + \frac{3}{2} m_{W1} &= \frac{6}{5} \frac{3}{2} m_{A1} + \frac{6}{5} \frac{10}{9} m_{W1} \\ m_{A1} &= \frac{5}{9} m_{W1}. \end{aligned} \tag{9.4}$$

We divided both the equations by m_{A1}, which is definitely nonzero. Now, let's get back to the equation 9.3, where we can plug $m_{A1} = \frac{5}{9} m_{W1}$ and see that the result is

$$\frac{\frac{5}{9} m_{W1}}{\frac{5}{9} m_{W1} + m_{W1}} = \frac{5}{6} X \quad \Rightarrow \quad X = \frac{3}{7} \doteq 42.86\ \%. \tag{9.5}$$

10 First we need to realize that the origins of all torques are the plane of a chandelier. While all five candles are on the chandelier, all torques are balanced and the chandelier is in equilibrium. Once a candle falls off, the symmetry of the torques is disturbed and the chandelier will rotate until its centre of mass has minimal altitude. That means that the empty arm will move to the highest possible point, which occurs when the angle between the plane of the chandelier and the horizontal plane is 90°.

11 If we kept pouring sand in one place, we would create a cone with surface slanted at an angle β. By taking the union of all such cones that can fit onto the parcel we obtain a shape shown in figure 11.1.

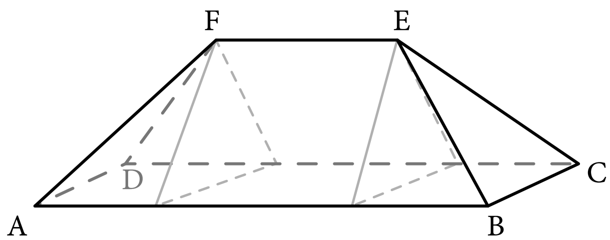


Figure 11.1: *Side view*

This is a triangular prism truncated on both sides. We can split this shape into three parts – a triangular prism and two half-pyramids that can be merged into a single pyramid with sides slanted at angle β. Let us first calculate its height. Slicing the pyramid with a plane perpendicular to the base creates an isosceles triangle with base b and two equal angles β. The height of the triangle is then

$$h = \frac{b}{2}\tan\beta. \tag{11.1}$$

Since all faces of the pyramid form an angle β with the horizontal plane, it must be a right pyramid and its base must be a square with side length b. With this knowledge we can easily deduce that the triangular prism has length $d = a - b$ and its volume is thus

$$V_h = S_p \cdot d = \frac{hb}{2}d = \frac{b^2}{4}(a-b)\tan\beta. \tag{11.2}$$

The volume of the pyramid can be calculated easily as

$$V_i = \frac{1}{3}S_p h = \frac{b^3}{6}\tan\beta \tag{11.3}$$

and the maximum volume of sand is therefore

$$V = V_h + V_i = \left(\frac{3a-b}{12}\right)b^2\tan\beta. \tag{11.4}$$

$\boxed{12}$ Average speed $\overline{v}(t)$ at any given time t is calculated as the distance travelled during the time interval t divided by the time t, in our case since $t = 0$. Therefore

$$\overline{v}(t) = \frac{s(t)}{t}. \tag{12.1}$$

In other words, we can choose any time t in the graph and calculate the distance that has been travelled until then. The we plug these values into the formula for average speed. If we do this for all values of t, we can simply find the maximum of $\overline{v}(t)$. It can, however, be easily seen from the graph in the picture 12.1. Let us modify the formula for average speed so that t is the independent variable:

$$s(t) = \overline{v}(t)t. \tag{12.2}$$

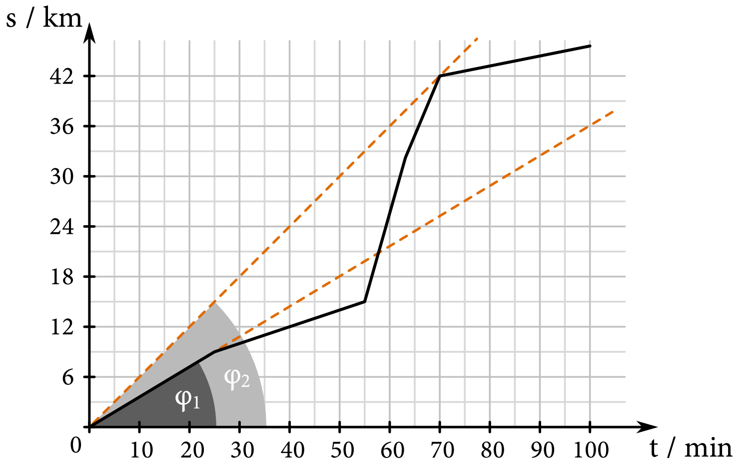


Figure 12.1: *Angle φ_1 is smaller than φ_2 which means that $\overline{v}(t_1) < \overline{v}(t_2)$.*

Now we can see that $\overline{v}(t)$ is the slope of the line in the graph. Thus, if we want to find the maximum of $\overline{v}(t)$, we only need to find the line with the steepest slope, that is, the line for which the angle from the x axis is the largest. We can see in the graph that this occurs at $t = 70$ min and it is exactly

$$\overline{v}(t) = \frac{42}{70}\ \text{km/min} = 36\ \text{km/h}. \tag{12.3}$$

13 Since we can distinguish the guests by their names, there are $5! = 120$ possible permutations of how the tea cups can be assigned. This looks difficult, but we can quickly eliminate a lot of them.

To keep the seesaw balanced, the moments of force from both sides must be equal. Let's denote the saucers k_{-2}, k_{-1}, k_0, k_{+1} a k_{+2} and assign the weights 1, 2, 3, 4 and 5 to them. If the distance between two saucers is w and the mass of the smallest one is m, in the equation describing moments of force w, m and g cancel out and we are left with the equation

$$\begin{aligned} k_{-2}mg2w + k_{-1}mgw = k_{+1}mgw + k_{+2}mg2w \qquad &\Big/ /mgw \\ 2k_{-2} + k_{-1} = k_{+1} + 2k_{+2}. \end{aligned} \tag{13.1}$$

At a distance w from the centre the masses of cups must have the same parity – if k_{-1} i is odd, then k_{+1} is also odd and vice versa; otherwise, the moments of force from the left and right side would have different parity. This cannout be balanced out by any combination of cups at $k_{\pm 2}$. And the final observation – to every solution we can unambiguously assign its mirror image.

Starting with the heaviest cup, we need to examine three cases:

- If we put it to k_{-2}, at k_{+2} we need to place the cup with mass $4m$ – otherwise, we wouldn't be able to balance the seesaw. From the parity condition, at $k_{\pm 1}$ we need to place the cups with masses m and $3m$, and the solution is determined as $5:1:2:3:4$ or $4:3:2:1:5$.
- If we put the heaviest cup to k_{-1}, from the parity condition we know that k_1 must be occupied by a cup whose mass is an odd multiple of m. That means we have two options:

  - if we put the cup with mass m there, the difference between $k_{\pm 2}$ must be $2m$, which results in solutions $2:5:3:1:4$ and $4:1:3:5:2$;
  - if we put the cup with mass $3m$ there, the difference between $k_{\pm 2}$ must be m, which uniquely determines the solutions $1:5:4:3:2$ and $2:3:4:5:1$.

- Finally, if we place the heaviest cup onto the pivot, the parity condition leaves only two options for us,
  - if positions $k_{\pm 1}$ are occupied by cups with masses that are even multiples of m, we are unable to put m and $3m$ cups onto the ends and maintain balance;
  - and in the other case, positions $k_{\pm 1}$ are occupied by cups with masses that are even multiples of m, and again, we quickly find that we cannot balance the seesaw anymore.

We have analyzed all 120 permutations and found that only six of them are valid.

14 If we dumped all the dirt at one place, we would get a cone with slant angle β. In the image below we see figure we will get by taking the union of all such cones.

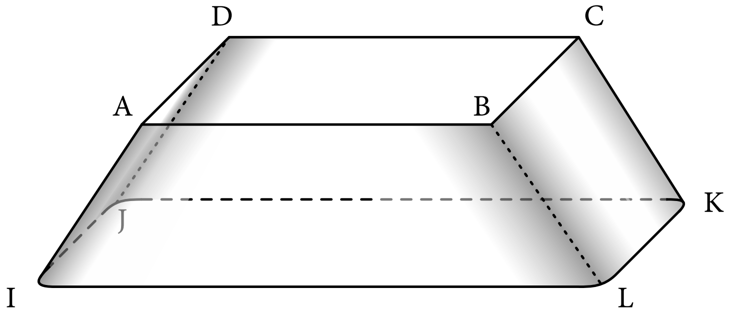


Figure 14.1: *Pohľad zhora*

Let s be the distance between the outer edge of our construction and the projection of the monument base. Since the slant angle is β everywhere, this distance will be the same everywhere. We can express it as

$$\cot\beta = \frac{s}{h} \quad \Rightarrow \quad s = h\cot\beta.$$

Now we need to realise that we can divide our figure into several parts. In the corners, we will get four quarters of a cone with radius s and height h. Then we have four orthogonal prisms which can be joined into one long prism. Its height is $2a+2b$ and its nase is a right triangle with sides h and s. The last part will be a cuboid with sides $a\cdot b\cdot h$. Their volumes are

$$V_{\text{cone}} = \frac{1}{3}\pi s^2 h = \frac{1}{3}\pi h^3\cot^2\beta,$$

$$V_{\text{cuboid}} = abh,$$

$$V_{\text{prisms}} = S_p(2a+2b) = \frac{hs}{2}(2a+2b) = \frac{h^2}{2}(2a+2b)\cot\beta$$

and the final volume is

$$V = V_{\text{cone}} + V_{\text{cuboid}} + V_{\text{prisms}} = \frac{1}{3}\pi h^3\cot^2\beta + abh + h^2(a+b)\cot\beta.$$

After plugging in the values we find out that William will need approximately 391.27 m^3 of soil.

15 If there are two cogwheels touching each other, they must have equal tangential speeds, but opposite directions of rotation. This means that all small gears will have equal tangential speeds (same as the tangential velocity of the bigger wheel), which is also the speed with which the belt moves. We can express it as 36 cogs per one rotation of the bigger wheel. Let us denote the with of one cog as d. Then one rotation of the large cogwheel corresponds to a movement of the belt by $36d$. The only thing we need to calculate now is the length of the belt.

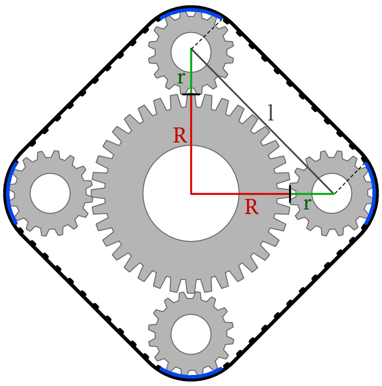


Figure 15.1: *Geometry of the belt*

In the picture we can see that the belt consists of four arc of 90° which we can put together to make one full circle with circumference $16d$. The rest of the belt consists of four line segments of length l, which is also the distance between the centers of two adjacent small wheels. We can calculate it as the diagonal of a square with side length $R + r$, where R is the radius of the bigger wheel, or $R = \frac{36d}{2\pi}$, and r is the radius of the smaller wheel, or $r = \frac{16d}{2\pi}$. Each of the four line segments then has length

$$\sqrt{2}(R + r) = \frac{26\sqrt{2}}{\pi}d. \tag{15.1}$$

The length of the belt is $\left(16 + 104\frac{\sqrt{2}}{\pi}\right)d$ and in one revolution of the bigger wheel it moves by $36d$. It will therefore take

$$\frac{\left(16 + 104\frac{\sqrt{2}}{\pi}\right)d}{36d} = \frac{4}{9} + \frac{26\sqrt{2}}{9\pi} \doteq 1.745 \tag{15.2}$$

revolutions of the large cogwheel for the belt to make one revolution.

Let us just add, that if the belt has teeth as well, then its length must be an integer multiple of the width of one cog. Furthermore, if we want all gears to work equally hard, then this integer multiple must be divisible by four. In that case we find that the length of the belt must be $64d$, therefore the bigger gear must rotate $\frac{64d}{36d} \doteq 1.78$-times.

16 The atmospheric pressure on Mars is much lower than on the Earth, so the barometer should display a much lower value. However, we must not forget that the barometer measures the relationship between the atmospheric pressure and the hydrostatic pressure in mercury – which also depends on the magnitude of acceleration due to gravity which is different on Mars.

The gravitational acceleration is given by

$$g_{♂} = \frac{GM_{♂}}{R_{♂}^2}, \tag{16.1}$$

and on small scales the hydrostatic pressure is

$$p = \rho g h, \tag{16.2}$$

so we can express

$$h = \frac{pR_{♂}^2}{\rho G M_{♂}}. \tag{16.3}$$

After plugging in the numerical values from the table of constants or other sources we find out that the height of the mercury column is about 12 mm.

17 To calculate the work we need to know the magnitude of the force F stretching the string. Let us draw the picture 17.1. Let the string be inclined from the vertical by angle α. Two forces are now acting on the point mass – force $\vec{F}$ from the string and the force of gravity mg. The point is moving with angular velocity ω on a circle with radius $\ell \sin\alpha$.

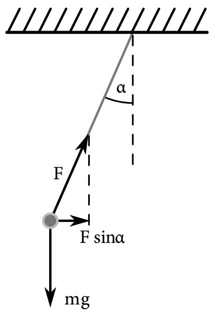


Figure 17.1: *Forces acting on the point mass.*

To keep it on a circular trajectory, some centripetal force with magnitude $m\omega^2\ell\sin\alpha$ must be act on the point mass. The only force with a horizontal component is the force F coming from the string, so we may write the equation

$$F\sin\alpha = m\omega^2\ell\sin\alpha \quad \Rightarrow \quad F = m\omega^2\ell. \tag{17.1}$$

The work Wihlelmina does while pulling the string by a small distance $\Delta\ell$ is then simply

$$\mathrm{d}W = m\omega^2\ell\,\Delta\ell. \tag{17.2}$$

18 At first, let is calculate the dimensions of the flag and use them to find the resistances of the luminous strips. To find the lengths we need nothing more complicated than the Pythagorean theorem. By multiplying the lengths by resistivity 30 Ω/m we obtain their respective restistances, which are displayed in the picture 18.1.

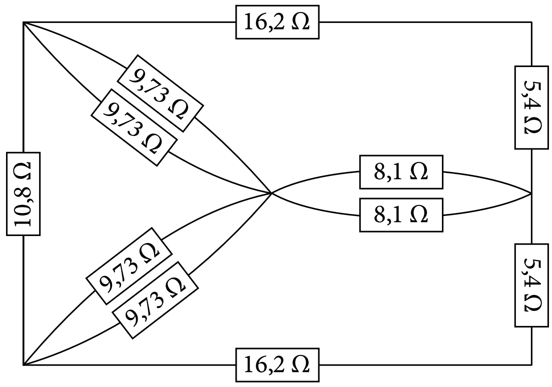


Figure 18.1: *Luminous strips as resistors.*

The pair of strips between the white and the red part is interesting – on both of their ends, the potentials are equal. How do we prove this? We could use a simple trick – if we mirror the entire circuit so that the supplying wires are displayed onto each other and the circuit does not change, the points that are displayed onto themselves must have equal potentials. Now, you may ask, is this fact useful for us? Yes, it is! If two points are equal potentials, no current flows between them, even if there is a wire. Hence we may cut the wire and it will not affect the circuit in any way.

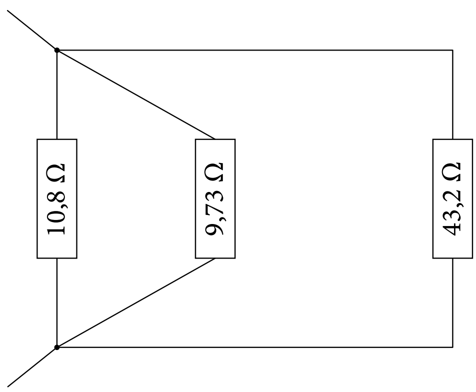


Figure 18.2: *Wires with current, after their resistance was calculated.*

Now we are left with a combination of resistors connected only in series or in parallel. The voltage drop on every single branch will be equal to the supply voltage, 42 V. We do know their resistances and that means the current in each of the branches can be calculated using Ohm's law, $I = \frac{U}{R}$. The branch with resistance of 9.73 Ω is composed from two equal parallel branches; so half of the current will flow through each of them. From left to right, the currents in the branches are 3.89, 2.16, 2.16 and 0.97 A. Currents higher than 2 A – currents blowing a fuse – flow only in the branches displayed as thick lines.

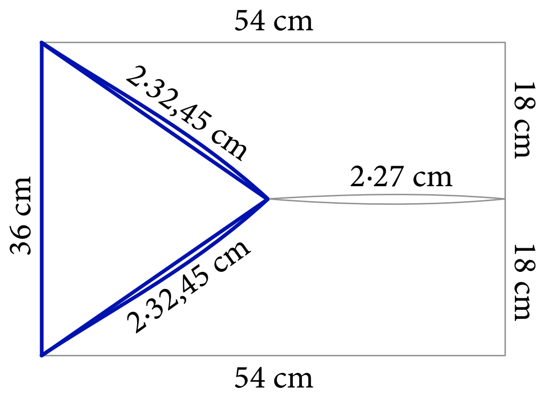


Figure 18.3: *Lenghts of luminous strips, thick lines do denote strips with current higher than 2 A.*

What is the probability that the technician cuts somewhere in these places? We can calculate that as the ratio of the length of these strips to the sum of all the strips' lengths,

$$P \approx \frac{165.8\ \text{cm}}{363.8\ \text{cm}} \approx 0.456 \doteq 46\ \%. \tag{18.1}$$

19 An emergency brake must slow the train down in the shortest possible distance, and that is determined by the coefficient of friction. THe mass of the train remains the same ($m = 100\ \text{t} + 5 \cdot 20\ \text{t} = 200\ \text{t}$), but the braking force can only come from the vehicles whose brakes are already active. It will be equal to $F = \mu m' g$ where m' is the total mass of the braking vehicles and μ is the coefficient of dynamic friction. The deceleration of the train is then

$$a = \frac{F}{m} = \frac{\mu m' g}{m}. \tag{19.1}$$

When Lucy activates the brakes, the deceleration will be

$$a_0 = \frac{10^5\ \text{kg}}{2 \cdot 10^5\ \text{kg}} \cdot 0.2 \cdot 10\ \text{m/s}^2 = 1\ \text{m/s}^2. \tag{19.2}$$

After each second, it increases by extra

$$\Delta a = 0.2 \cdot \frac{20\ \text{t}}{200\ \text{t}} \cdot 10\ \text{m/s}^2 = 0.2\ \text{m/s}^2 \tag{19.3}$$

until it reaches its maximal value $a_{\text{max}} = a_0 + 5\,\Delta a = 2\ \text{m/s}^2$ and then train will decelerate at this rate until it comes to a halt. This can be readily expressed in mathematical formula, but during the competition it is probably easier and faster to draw everything (see picture 19.1) and write into the table 19.1.

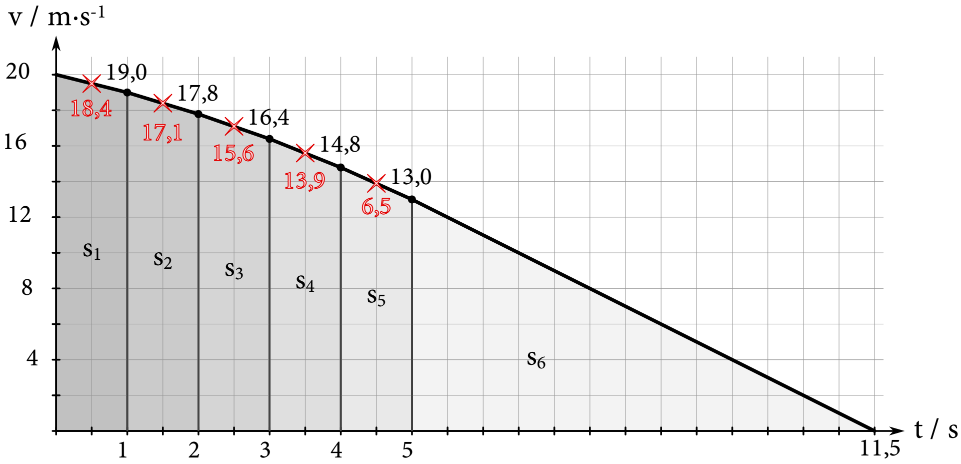


Figure 19.1: *The speed of the train as a function of time*

In each interval the train uniformly decelerates from speed v_i to a lower speed $w_i = v_i - a_i\,\Delta t_i$, whicih means we can express its average speed in this interval as the mean of the speeds at its beginning and its end. Furthermore, $v_{i+1} = w_i$, since its speed has to be continuous. The last interval is special: all vehicles are braking and the train will slow down until it stops. From the speed of $v_5 = 13$ m/s and at a rate of 2 m/s^2 this will take 6.5 s and the travelled distance will be $s_5 = \frac{169}{4}$ m = 42.25 m. Finally we express the distance travelled in each interval as the product of its length and the train's average speed in it, $s_i = \overline{v}_i \cdot \Delta t_i$.

Table 19.1: Instantaneous decelerations and the corresponding distance.

number of braking vehicles	Δt_i / s	F_i / kN	a_i / m/s^2	v_i / m/s	w_i / m/s	$\overline{v}_i$ / m/s	s_i / m
0	1	200	1.0	20.0	19.0	19.5	19.50
1	1	240	1.2	19.0	17.8	18.4	18.40
2	1	280	1.4	17.8	16.4	17.1	17.10
3	1	320	1.6	16.4	14.8	15.6	15.60
4	1	360	1.8	14.8	13.0	13.9	13.90
5	6.5	400	2	13.0	0.0	6.5	42.25

Now we only need to sum everything and we find out that the total braking distance is 126.75 m.

20 First we need to realize, that the blocks must be oriented horizontally, because otherwise if the blocks were tilted at an angle, then due to low friction, the blocks would start slipping and they would fall into the pit. Thus the solution will consist of two blocks placed on the floor, each one protruding into the gap by length x. The third block will be placed on these two blocks in such a way, that it will only be supported by their edges (see figure 20.1).

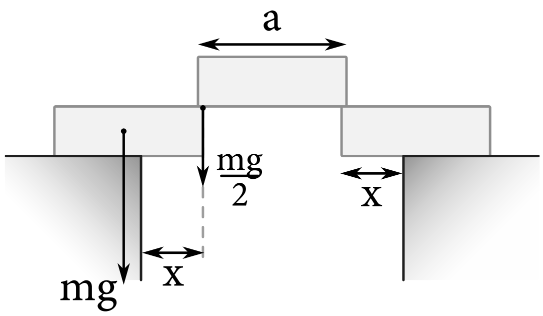


Figure 20.1: *The bridge with maximal span.*

Thus we need to figure out the maximum distance x by which the blocks can protrude over the gap. For the blocks to not fall into the pit, the total torque produced by forces acting on the lower blocks must be zero. Let us choose one of these lower blocks and for the axis of rotation, we will choose the edge of the gap. The gravitational force acts in the block's center of mass, which is in distance $a/2 - x$ from the axis, and the upper block acts with force $mg/2$ in the edge of the block, at distance x from the axis of rotation. This yields

$$mg\left(\frac{a}{2} - x\right) = \frac{mg}{2}x. \tag{20.1}$$

From this we see that $x = a/3$ and the total span of the bridge is

$$d = 2x + a = \frac{5}{3}a. \tag{20.2}$$

21 FIndin the minimal distance between two points on a cone may seem difficult, but it is not so, since its surface can be unrolled into a flat plane. The base of the cone is not important in this case and we are left with a disk sector.

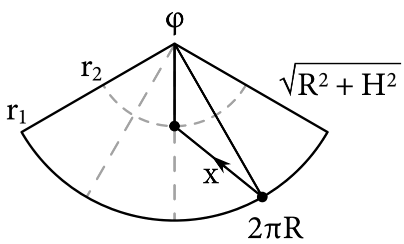


Figure 21.1: *Surface development of the cone.*

As the radius of the base is R, the curved part of its circumference must be $2\pi R$ long. Its radius is $\sqrt{R^2 + H^2}$, as it is the hypotenuse of a right triangle with sides R (the radius of the cone's base) and H (its height). The angle at the apex of the disk sector is $\varphi = \frac{2\pi R}{\sqrt{R^2+H^2}}$ and the ant wants to traverse 90° on the cone, which means it will need to travel an angle of $\varphi/4$ on the developed surface.

On a plane the shortest route is obviously a straight line. If we connect its endpoints with the apex, we obtain a triangle where we can apply the law of cosines. We already know the central angle $\varphi/4$, the distance to the beginning is obviously $r_1 = \sqrt{R^2 + H^2}$ and the distance to the endpoint is $r_2 = \frac{1}{2}\sqrt{R^2 + H^2}$ since it scales linearly with altitude and the ant wants to get halfway up.

Now let us denote the unknown distance x. From the cosine law

$$x^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\frac{\varphi}{4}, \tag{21.1}$$

which, after substituting for r_1, r_2 and φ yields

$$x = \sqrt{R^2+H^2}\sqrt{\frac{5}{4} - \cos\frac{\pi R}{2\sqrt{R^2+H^2}}}. \tag{21.2}$$

After plugging in the physical dimensions we see that $x \doteq 1.56$ m.

22 Forces applied on a stick are gravitational force pointing downwards and electric forces from other electric charges pointing in different horizontal directions. And there is of course normal force from ceiling, so the whole system is in the state of equilibrium. However, this force is unknown, therefore we need to find other way to solve this problem. The entire system is stationary, that means the sticks aren't rotating and we need to zero the torques.

Let us denote electric charge as q, mass of the stick m and its length d. From symmetry of the problem, we can assume that the sticks create a pyramid with a square base with a side length r and diagonal length $\sqrt{2}r$. We can choose any of the four sticks and write equations with forces from nearest charges. Both electric charges have the magnitude

$$F_{e1} = \frac{q^2}{4\pi\varepsilon_0 r^2} \tag{22.1}$$

and they are proportional to each other. Therefore, magnitude of the net force is $\sqrt{2}F_{e1}$ and vector of net force is oriented from the centre of the square to the charge, where te force is applied. Magnitude of the electric force from the farthest charge is

$$F_{e2} = \frac{q^2}{4\pi\varepsilon_0(\sqrt{2}r)^2}. \tag{22.2}$$

So the total net force applied on the charge has magnitude $F_e = \sqrt{2}F_{e1} + F_{e2}$ and it is oriented from the centre of the square. Gravitational force is oriented downwards and it is applied in the centre of mass of the stick as shown in the figure 22.1.

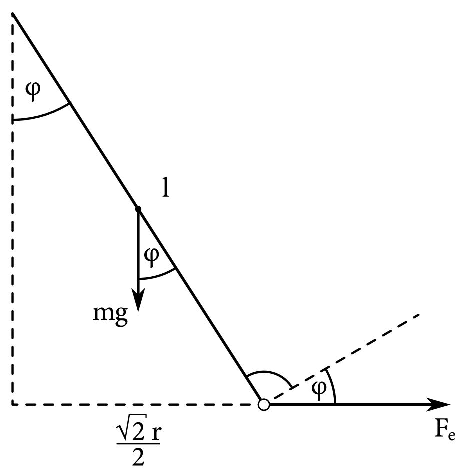


Figure 22.1: *Sily pôsobiace na jednu paličku.*

The botom segment of the triangle is half of the square diagonal, so its length is $\frac{\sqrt{2}r}{2}$. We can also notice from the figure that $r = \frac{2}{\sqrt{2}} d \sin\varphi$.

$$
\begin{aligned}
mg\frac{d}{2}\sin\varphi &= \frac{q^2}{4\pi\varepsilon_0 r^2}\left(\frac{1}{2}+\sqrt{2}\right)d\cos\varphi \\
mg\frac{d}{2}\sin\varphi &= \frac{q^2}{4\pi\varepsilon_0\left(\frac{2}{\sqrt{2}}d\sin\varphi\right)^2}\left(\frac{1}{2}+\sqrt{2}\right)d\cos\varphi \\
\frac{\sin^3\varphi}{\cos\varphi} &= \frac{q^2}{4\pi\varepsilon_0 d^2 mg}\left(\frac{1}{2}+\sqrt{2}\right).
\end{aligned}
\tag{22.3}
$$

At this point we see that this problem doesn't have a closed-form solution, so we can't express φ directly. However, we can use calculator and try to guess (wisely) the answer. With a quick binary search we should find out that the answer is $\varphi \doteq 68°$.

23 Let us denote the mass of the lorry M and the mass of the fly m. Let the velocity of the lorry be v, the velocity of the fly $-v$ and the velocity of both of them after the collision u. The law of conservation of momentum gives

$$
Mv - mv = (M+m)u \quad \Rightarrow \quad u = \frac{M-m}{M+m}v. \tag{23.1}
$$

The collision is inelastic so energy is not conserved. The difference between the energy before and after the collision is equal to the heat created by the collision

$$
\begin{aligned}
Q &= \frac{1}{2}M\left(v^2-u^2\right) + \frac{1}{2}m\left((-v)^2-u^2\right) \\
&= 2\frac{Mm}{M+m}v^2.
\end{aligned}
\tag{23.2}
$$

Mass of the lorry is, obviously, much larger than mass of the fly, therefore we can neglect m in the denominator. Therefore

$$
Q \approx 2mv^2. \tag{23.3}
$$

The problem statement says that all heat is spent to increase the former fly, so

$$
Q = mc\,\Delta t. \tag{23.4}
$$

After joining the last two equations, we obtain

$$
\Delta T \approx \frac{2v^2}{c} \approx 0.13\ \text{°C}. \tag{23.5}
$$

24 In the beginning, Matthew has zero velocity, therefore zero kinetic energy. However, when he gets to the point with velocity v, his potential energy decreases by mgr. Since we don't consider friction and drag force, his kinetic energy increases by this factor. His velocity is well-known, so we can find the radius from the law of conservation of energy

$$\frac{1}{2}mv^2 = mgr \quad \Rightarrow \quad r = \frac{v^2}{2g}. \tag{24.1}$$

When he gets to the lowest point of the loop his kinetic energy increases again by the mgr, so $mv^2 = 2mgr$. New velocity, we can call it w, will be $\sqrt{2}$ times greater. Therefore, we can express magnitude of centripetal force as

$$a_c = \frac{w^2}{r} = \frac{2v^2}{r} = \frac{2v^2 \cdot 2g}{v^2} = 4g. \tag{24.2}$$

Final acceleration is independent of the radius of the loop and his velocity v. Apart from centripetal force, Matthew also feels reaction from gravitational force with magnitude g, therefore total g-force in the lowest point is $5g$.

25 First, let us show what happens if we connect the springs in parallel. The resultant force from the springs is the sum of the forces from the individual springs. At the same time the extension of the individual springs is the same as the extension of the whole system, thus we get

$$\begin{aligned} F &= F_1 + F_2 \\ k \cdot \Delta x &= k_1 \cdot \Delta x + k_2 \cdot \Delta x \\ k &= k_1 + k_2. \end{aligned} \tag{25.1}$$

We can see that if we have two identical springs in parallel, they act like one spring with double the stiffness. If we have n springs, they will act like a single spring that is n times stiffer.

We know from the statement that each spring has an unknown rest length, which we will denote as s. We also know that when Joe hangs on one spring, his distance from the ground is 50 cm – the extension of the spring Δx_1 together with the spring's rest length s is 200 cm. When he hangs on two springs, the extension of the springs Δx_2 together with their rest length s is 110 cm (because it is 140 cm above the ground). And when he hangs on three springs, the extension of the springs is Δx_3. Together with the rest length s we get the desired distance which we can label as ψ.

With this notation, we get the equations of force,

$$\begin{aligned} F = \quad mg = \quad k\,\Delta x_1 &= \quad 2k\,\Delta x_2 = \quad 3k\,\Delta x_3, \\ \Delta x_1 &= \quad 2\,\Delta x_2 = \quad 3\,\Delta x_3, \\ 200\text{ cm} - s &= 2(110\text{ cm} - s) = 3(\psi - s). \end{aligned} \tag{25.2}$$

From this we can easily calculate the rest length of the springs

$$200\ \text{cm} - s = 220\ \text{cm} - 2s \quad \Rightarrow \quad s = 20\ \text{cm}. \tag{25.3}$$

Once we know s, calculating the distance from the ceiling ψ is trivial

$$3\psi - 3s = 200\ \text{cm} - s \quad \Rightarrow \quad \psi = 80\ \text{cm}. \tag{25.4}$$

To find Joe's distance from the ground, we only need to subtract this from the total height of the room, so the result is $250\ \text{cm} - \psi = 170\ \text{cm}$.

26 Let's review our knowledge of the forces acting on a charged particle in an electric and in a magnetic field first. In the electric field, the acting force is the electric force

$$\overrightarrow{F_e} = Q\overrightarrow{E}, \tag{26.1}$$

where Q is the particle's charge; in a magnetic field, the acting force is the magnetic force

$$\overrightarrow{F_m} = Q\left(\overrightarrow{v} \times \overrightarrow{B}\right), \tag{26.2}$$

where $\overrightarrow{v}$ is the particle's velocity. The electric force is either in the direction of the particle or in the opposite direction, depending on the particle charge; the magnetic force is perpendicular to the magnetic field and the particle velocity vector. Concluding from these statements, the electric force can change the particle's speed, while the magnetic force can only alter its direction, i. e. it is a centripetal force.

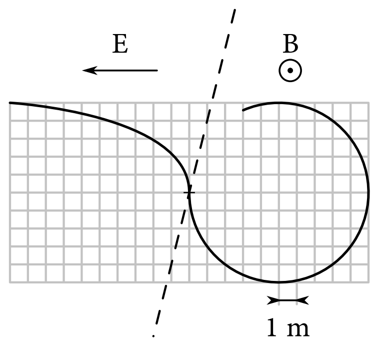


Figure 26.1: *The motion of the ball*

Let's analyze the motion of the ball. We will denote the horizontal coordinate as x and the vertical one as y. In the figure 26.1, the ball's trajectory is traced, but its direction is not marked. However, a simple algorithm is sufficient for its determination. Let's assume the ball first traverses the magnetic field, then enters the electric field. From the trajectory curvature in the magnetic field it is clear the ball's charge is positive. When a positively charged particle enters the electric field, the electric force and intensity are identically oriented. Therefore the ball is accelerated in the direction to the left, which is in accordance with our observation from 26.1, where the x coordinate augments together with the y coordinate. If the ball were to move in the opposite direction, from the traced trajectory in the electric field it would seem the ball is decelerating, so it would be positively charged. However, such an assumption would suggest the curvature should be opposite after the ball enters the magnetic field.

Let the ball's speed be v. The magnetic field is perpendicular to the trajectory plane, and it will always be perpendicular to the velocity vector, therefore the magnitude of the magnetic force is

$$F_m = QvB. \tag{26.3}$$

The magnetic force is centripetal, therefore

$$QvB = \frac{mv^2}{r} \quad \Rightarrow \quad \frac{Q}{m} = \frac{v}{Br},$$

where m is the ball's mass. Since m is constant, the ball's trajectory is a circle. From the 26.1, its radius is $r = 5$ m. Again, from the direction of the curvature, the charge of the ball is positive.

The ball transits from the magnetic to the electric field. There cannot be any non-uniformity in the change of the velocity vector, as this would mean an infinite force acting in this point. From the figure we see that once the ball enters the electric field, it moves in the direction of the y coordinate with the same speed it had in the magnetic field, v. Let's assume the ball stays in the electric field for a time interval t, during which it traverses a distance of $\Delta y = 5$ m in direction y, and a of distance $\Delta x = 10$ m in the direction x. There is no acting force in the y direction, therefore

$$\Delta y = vt;$$

in the x direction, the acting force is

$$F_e = QE,$$

which describes a uniformly accelerated motion with acceleration

$$a = \frac{QE}{m}$$

and with a zero initial velocity. Therefore

$$\Delta x = \frac{1}{2}\frac{QE}{m}t^2. \tag{26.4}$$

After eliminatng time from these equations we obtain

$$\Delta x = \frac{1}{2}\frac{QE}{m}\left(\frac{\Delta y}{v}\right)^2,$$

in which we recognize a parabola.

Substituting $\frac{Q}{m}$, we get

$$\Delta x = \frac{1}{2}\frac{v}{Br}E\left(\frac{\Delta y}{v}\right)^2 \quad \Rightarrow \quad v = \frac{1}{2}\frac{E}{B}\frac{(\Delta y)^2}{r\,\Delta x}. \tag{26.5}$$

For the numerical values given in the problem statement and seeing the figure 26.1, we can calculate that $v = 0.5$ m/s.

27 A full hippo is a solid ball with radius r and mass m. The moment of inertia of a ball is

$$I_1 = \frac{2}{5}mr^2. \tag{27.1}$$

What about the hungry hippo? Its shape is a sphere (a full hippo) without a stomach. To find the moment of inertia, we'll have to subtract the moment of inertia of the stomach from the previous result. The stomach is a sphere with mass $m/8$, so its radius is $r/2$. The moment of inerta of the second hippo is then

$$I_2 = \frac{2}{5}mr^2 - \frac{2}{5}\frac{m}{8}\left(\frac{r}{2}\right)^2 = \frac{31}{80}mr^2. \tag{27.2}$$

Now, let's think of the energy conservation law! Let the hippos lie at an altitude h over the Nile's water level; that does mean that during their roll the hippos have to descent by h. The potential energy of the full hippo will decrease by mgh; this energy is going to transform into rotational and translational parts of hippo's kinetic energy. Angular speed can be determined as $\omega = v/r$, because the hippo is not slipping. For the full hippo we obtain

$$\begin{aligned} mgh &= \frac{1}{2}I_1\left(\frac{v_1}{r}\right)^2 + \frac{1}{2}mv_1^2 \\ &= \frac{1}{5}mv_1^2 + \frac{1}{2}mv_1^2 \\ &= \frac{7}{10}mv_1^2, \end{aligned} \tag{27.3}$$

from which we can express

$$v_1 = \sqrt{\frac{10}{7}gh}. \tag{27.4}$$

Then we do the same for a hungry hippo. We have to remember that this hippo's mass is only $\frac{7}{8}m$. Therefore

$$\begin{aligned} \frac{7}{8}mgh &= \frac{1}{2}I_2\left(\frac{v_2}{r}\right)^2 + \frac{1}{2}\frac{7}{8}mv_2^2 \\ &= \frac{31}{160}mv_2^2 + \frac{7}{16}mv_2^2 \\ &= \frac{101}{160}mv_2^2 \end{aligned} \tag{27.5}$$

and from that

$$v_2 = \sqrt{\frac{140}{101}gh}. \tag{27.6}$$

The ratio of these speeds is

$$\frac{v_1}{v_2} = \frac{\sqrt{202}}{14}, \tag{27.7}$$

with the full hippo being faster.

28 When the ball is being submerged into the lake, the hydrostatic pressure slowly rises and compresses the ball. When the ball shrinks so much that its average density exceeds the density of water, the weight of the ball definitively overtakes the buoyant force and the ball will no longer emerge on its own. Let us denote the mass of the empty ball $m = 0.1\ \mathrm{kg}$, R_0 its radius and $V_0 = \frac{4\pi}{3}R_0^3 \approx 0.0042\ \mathrm{m^3}$ its volume outside the water.

When Lucy inflates the ball, its total mass increases by the mass of the air to

$$M = m + V_0\rho_a, \tag{28.1}$$

where ρ_a is the density of air at standard temperature and pressure. Since Lucy submerges the ball very slowly, the temperature of the gas remains equal to the temperature of water during the entire process. Thus we have an isothermal process, where the equation $pV = \text{const}$ holds.

The pressure at the water surface is just the atmospheric pressure p_0. In depth h under the surface, this pressure increases by hydrostatic pressure $\rho_w gh$, where ρ_w is the density of water. Let us denote the volume of the ball in depth h as V. Then the isothermal process yields

$$p_0V_0 = (p_0 + \rho_w gh)V. \tag{28.2}$$

From this equation we can easily express the volume of the ball in depth h as

$$V = V_0\frac{p_0}{p_0 + \rho_w gh}. \tag{28.3}$$

If we want to find the critical depth, we need to set the average density of the ball equal to the density of water, therefore

$$\frac{M}{V} = \rho_w. \tag{28.4}$$

Now we only need to plug the mass M from equation 28.1 and volume V of the ball from equation 28.3 into this to get

$$\frac{m + V_0\rho_a}{V_0\frac{p_0}{p_0+\rho_w gh}} = \rho_w \tag{28.5}$$

and express the depth h. The answer is

$$h = \frac{p_0}{\rho_w g}\left(\frac{4\pi R_0^3\rho_w}{3m + 4\pi R_0^3\rho_a} - 1\right) \approx 392\ \text{m}. \tag{28.6}$$

29 Since there is no momentum of force acting on the system from the outside of it, the angular momentum is conserved. The energy is not conserved since there is friction and thus some energy is converted into heat. If I_1 is moment of inertia of the disc, the initial angular momentum is $L = I_1\omega$.

After the square with moment of inertia I_2 starts rotating, the total moment of inertia is I_1+I_2 and the angular frequency drops to ω'. The angular momentum is therefore $L = (I_1+I_2)\omega'$. The law of conservation of angular momentum gives

$$\omega' = \frac{I_1}{I_1 + I_2}\omega. \tag{29.1}$$

The remaining part is to plug in the moments of inertia. Moment of inertia of a homogeneous disc $I_1 = \frac{1}{2}m_1r^2$ which is $I_1 = \frac{\pi}{2}\sigma r^4$ when expressed with area density. Moment of inertia of a homogeneous square rotating around its centre can be determined using the parallel axis theorem and the fact that a square comprises of four smaller squares rotating around their corners. The moment of inertia of a square is $I_2 = Km_2a^2 = K\sigma a^4$

with an unknown constant K. It is an additive physical property so it can be expressed as a sum of moments of inertia of four smaller squares with side $\frac{a}{2}$ and mass $\frac{m}{4}$ and the rotation axis in the distance $l = \frac{a}{2\sqrt{2}}$ from their centre of mass. Their constant K is still the same because they are squares. Using the parallel axis theorem we obtain

$$
\begin{aligned}
Km_2a^2 &= 4\left(K\frac{m_2}{4}\left(\frac{a}{2}\right)^2 + \frac{m_2}{4}l^2\right), K\sigma a^4 = 4\left(K\sigma\left(\frac{a}{2}\right)^4 + \sigma\left(\frac{a}{2}\right)^2\left(\frac{a}{2\sqrt{2}}\right)^2\right),\\
\frac{3}{4}K\sigma a^4 &= \frac{1}{8}\sigma a^4,\\
K &= \frac{1}{6}.
\end{aligned}
\tag{29.2}
$$

Moment of inertia of the square is therefore $I_2 = \frac{1}{6}\sigma a^4$ and the final angular frequency is

$$
\omega' = \frac{\frac{\pi}{2}\sigma r^4}{\frac{\pi}{2}\sigma r^4 + \frac{1}{6}\sigma a^4}\omega = \frac{3\pi r^4}{3\pi r^4 + a^4}\omega. \tag{29.3}
$$

30 When electric current flows through a resistor, its temperature increases due to Joule heat. Heat from the resistor is flowing to surrounding air and we know that this heat is proportional to the temperature difference between the resistor and surroundings. The power of the resistor is

$$
P_+ = UI = \frac{U^2}{R}. \tag{30.1}
$$

and the power of heat flowing away is

$$
P_- = kS\,\Delta T, \tag{30.2}
$$

where S is the area of resistor and ΔT is the temperature difference. After a long time, the temperature of the resistor is constant, which means that all heat produced by resistor is flowing to the surroundings, so we can write $P_+ = P_-$

$$
\frac{U^2}{R} = kS\,\Delta T \quad \Rightarrow \quad \Delta T = \frac{U^2}{RkS}. \tag{30.3}
$$

Moreover, we can express the resistance of a conductor with length ℓ and cross-sectional area S as

$$
R = \rho\frac{\ell}{S}, \tag{30.4}
$$

where ρ is the resistivity of material. If we increase the proportions of the conductor by factor α, ℓ also increases by factor α and S increases by α^2. This scaling is independent of the shape of the conductor, that means if we had a conductor with different cross-section areas, every cross-section area would still scale by factor α. Therefore, we can think of the resistor as of an imperfect conductor. If we scale its size by factor α, its resistivity always decreases by α.

After a long time, the scaled resistor reaches its equilibrium temperature T_2 and the temperature difference with respect to the surrounding medium is $\Delta T_2 = T_2 - T_0$. With this knowledge we can simplify equation

30.3

$$\Delta T_2 = \frac{U^2}{\frac{R}{\alpha}k\alpha^2 S} = \frac{1}{\alpha}\frac{U^2}{RkS} = \frac{1}{\alpha}\Delta T. \tag{30.5}$$

Moreover, we know that $\alpha = 2$, so $\Delta T_2 = \frac{1}{2}\Delta T$. The temperature difference for the small resistor is $\Delta T = T_1 - T_0$, and for the big resistor it is $\Delta T = T_2 - T_0$, from where we can express

$$T_2 = \frac{T_0 + T_1}{2}.$$

31 The moment of inertia of the star can be calculated as the sum of all the moments of inertia from the individual rods about the axis passing through the centre of the star. The moment of inertia of a single rod with mass m and length d about its center of mass is $I_0 = \frac{1}{12}md^2$. If we want to calculate the moment of inertia about a different axis, we can use the parallel axis theorem, which states that if the new axis is parallel to the original axis, the new moment of inertia can be calculated as

$$I = I_0 + mx^2, \tag{31.1}$$

where x is the distance between the two axes.

In case of the star, the distance between the axis passing through the center of the rod and the axis passing through the center of the star is one third of the length of the median of the equilateral "triangles" that the star is made out of. The length of the median of such a triangle can be determined using the Pythagorean theorem as

$$a = \sqrt{d^2 - \left(\frac{d}{2}\right)^2} = \frac{\sqrt{3}}{2}d. \tag{31.2}$$

The moment of inertia of the star about its center is then

$$I = 6\left(I_0 + m\left(\frac{a}{3}\right)^2\right) = 6\left(\frac{1}{12}md^2 + \frac{1}{12}md^2\right) = md^2. \tag{31.3}$$

32 The stiffness of the spring in Dave's work of art can be calculated using the method of virtual work. It states that if the system is displaced from its equilibrium position by an infinitesimal amount, its energy does not change. The work of art is symmetric around the vertical axis, therefore it is enough if we look on one of its halves, the other must remain symmetric.

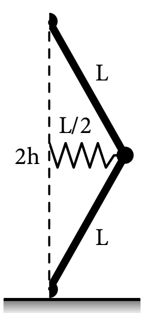


Figure 32.1: *Proportions of the "artwork" in equilibrium position*

Dave's system is in its equilibrium when the spring is stretched to length L. Then we can calculate the height of the system (let us denote it as $2h$) using the Pythagorean theorem

$$L^2 = \left(\frac{L}{2}\right)^2 + h^2, \quad \Rightarrow \quad h = \frac{\sqrt{3}}{2}L. \tag{32.1}$$

The total potential energy of the system can be calculated as the sum of gravitational potential energies of the four sticks and the potential energy of the spring with stiffness k. In equilibrium, the center of mass of the two sticks is at height $\frac{h}{2}$ and for the other two sticks at height $\frac{3h}{2}$. Therefore the energy is

$$E = 2\left(\frac{1}{2}mgh + \frac{3}{2}mgh\right) + \frac{1}{2}kL^2 = 4mgh + \frac{1}{2}kL^2. \tag{32.2}$$

Now suppose that the spring is stretched by an infinitesimal distance $\mathrm{d}x$. Since the length of the sticks cannot change, the height h will change to $h + \Delta h$, while the Pythagoras theorem must still hold:

$$L^2 = \left(\frac{L + \mathrm{d}x}{2}\right)^2 + (h + \mathrm{d}h)^2. \tag{32.3}$$

Neglecting the infinitesimal changes of second order, i. e. $(\mathrm{d}x)^2$ and $(\mathrm{d}h)^2$, yields a relation between the height and the length of the spring

$$\mathrm{d}h = -\frac{L}{4h}\,\mathrm{d}x. \tag{32.4}$$

Now let us look on the energy, where we are again neglecting the infinitesimal changes in second order

$$E = 4mg(h + \mathrm{d}h) + \frac{1}{2}k(L + \mathrm{d}x)^2 \approx 4mg(h + \mathrm{d}h) + \frac{1}{2}k(L^2 + 2L\,\mathrm{d}x). \tag{32.5}$$

Since the energy cannot change, we get

$$0 = 4mg\,\mathrm{d}h + kL\,\mathrm{d}x = \left(-mg\frac{L}{h} + kL\right)\mathrm{d}x. \tag{32.6}$$

Rearanging and plugging the height h into the equation gives us the stiffness of the spring,

$$k = \frac{mg}{h} = \frac{2}{\sqrt{3}}\frac{mg}{L}. \tag{32.7}$$

$\boxed{33}$ Let us denote the significant states of the gas by numbers from 1 to 6, as shown in the figure 33.1. The lowest temperature of the gas is is in point 1 and the highest is in state 4. Let the temperature in state 1 be $T_1 = T$. Then we can calculate the temperature in any other point using the ideal gas law, hence for any of these significant states we know the volume, pressure, and temperature.

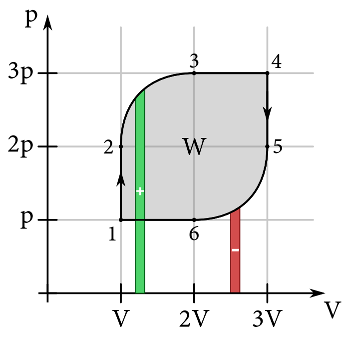


Figure 33.1: *pV diagram so zakresleným význačnými stavmi.*

We are interested in the efficiency of the engine. That is defined as the ratio of the work gained from the cycle to the total heat supplied into the gas

$$\eta = \frac{W}{Q_{\mathrm{in}}}. \tag{33.1}$$

Let us begin with the work. With a small change in volume ΔV the work done by gas equals $\Delta W = p\,\Delta V$. Geometrically, on the pV-diagram, it corresponds to the area of a small rectangle with width ΔV stretching from the curve to the V-axis. The total work done by the gas between two states is therefore equal to the area under the curve between these two states. The work between states 1 and 4 is positive, i. e. work is done by the gas, and the work between the states 4 and 1 is negative, which means that the work is done on the gas. The total work done by the gas is then equal to the area enclosed by the curve[2]

$$W = \left(2 + \frac{\pi}{2}\right) pV. \tag{33.2}$$

Now let's look on the supplied heat. The first law of thermodynamics tells us that the heat supplied to the system will be spent on work and change in internal energy

$$Q = W + \Delta U. \tag{33.3}$$

However, the internal energy is directly proportional to temperature. Each molecular degree of freedom contributes to the internal energy by amount $\frac{1}{2}kT$, and since all of these molecules have three degrees of freedom[3] and there are N molecules, so

$$\Delta U = \frac{3}{2} Nk\,\Delta T. \tag{33.4}$$

We only supply heat to the cycle between states 1 and 4. The temperature in state 4 is by the ideal gas law equal to

$$T_4 = \frac{p_4 V_4}{p_1 V_1} T_1 = \frac{3p}{p}\frac{3V}{V} T = 9T. \tag{33.5}$$

[2]Two rectangles $p \times V$ and two quarterellipses with semi axes p a V.
[3]The gas is monoatomic, therefore has three translational degrees and zero rotional.

The change in internal energy is then

$$\Delta U_{14} = \frac{3}{2}Nk(T_4 - T_1) = 12NkT \tag{33.6}$$

and the work done by the gas between these two states is

$$W_{14} = \left(5 + \frac{\pi}{4}\right)pV. \tag{33.7}$$

The total supplied heat is then

$$Q_{\text{in}} = W_{14} + \Delta U_{14} = \left(5 + \frac{\pi}{4}\right)pV + 12NkT. \tag{33.8}$$

Using the ideal gas law $pV = NkT$ we get

$$Q_{\text{in}} = \left(17 + \frac{\pi}{4}\right)pV. \tag{33.9}$$

If we plug this into the definition of efficiency, we get

$$\eta = \frac{\left(2 + \frac{\pi}{2}\right)}{\left(17 + \frac{\pi}{4}\right)} \doteq 0.2 = 20\ \%. \tag{33.10}$$

34 Geometrically, the problem is not that complicated, but the calculation is going to be a bit harder. From the picture, we see that for a real distance of two parsecs

$$2\ \text{pc} = \frac{2\ \text{au}}{\tan 1''} \tag{34.1}$$

and for Martin's unit $\mathcal{P}$

$$\mathcal{P} = \frac{1\ \text{au}}{\tan 0.5''}. \tag{34.2}$$

We are interested in the difference

$$\mathcal{P} - 2\ \text{pc} = \frac{1\ \text{au}}{\tan 0.5''} - \frac{2\ \text{au}}{\tan 1''}. \tag{34.3}$$

And here is the problem. $1''$ is a really small angle and the number of metres in one parsec is really huge. When subtracting two numbers that differ somewhere around the twelfth digit, the calculator rounds them in its memory and provides only a result which is really inaccurate, or it may even tell us it is zero.[4]

Let us use the Taylor expansion of the tangent function:

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \ldots \tag{34.4}$$

[4] A computer can do it. But you are not supposed to have one during Náboj. A good calculator may be able to calculate it too – if you have one, you're done here.

Considering the required accuracy of 1 km, a rough estimate tells us that using the first different term of expansion should be sufficient. Using this third-order term, we get

$$2\ \mathrm{pc} \approx \frac{1}{x+\frac{x^3}{3}} \cdot 2\ \mathrm{au} \qquad \text{and} \qquad \mathcal{P} \approx \frac{1}{\frac{x}{2}+\frac{\left(\frac{x}{2}\right)^3}{3}} \cdot 1\ \mathrm{au}, \tag{34.5}$$

where x is $1''$.

After some simplification, we obtain

$$\mathcal{P} - 2\ \mathrm{pc} \approx \frac{2}{x}\left(\frac{1}{1+\frac{x^2}{12}} - \frac{1}{1+\frac{x^2}{3}}\right) \cdot 1\ \mathrm{au}. \tag{34.6}$$

After these Taylor atrocities we will lose nothing if we also linearize

$$\frac{1}{1+x} \approx 1 - x, \tag{34.7}$$

which lets us simplify the equation 34.3 to

$$\mathcal{P} - 2\ \mathrm{pc} \approx 1\ \mathrm{au} \cdot \frac{2}{x}\left(1 - \frac{x^2}{12} - 1 + \frac{x^2}{3}\right) = 1\ \mathrm{au} \cdot \frac{2}{x}\frac{x^2}{4} = 1\ \mathrm{au} \cdot \frac{x}{2}. \tag{34.8}$$

Now we plug $x = 1'' = \frac{\pi}{180 \cdot 60 \cdot 60} = \frac{\pi}{648000}$ back into the equation and obtain the result

$$\mathcal{P} - 2\ \mathrm{pc} \approx \frac{1''}{2} \cdot 1\ \mathrm{au} = \frac{\pi}{1\,296\,000} \cdot 1.5 \cdot 10^8\ \mathrm{km} \doteq 364\ \mathrm{km}. \tag{34.9}$$

If we include the next term ($\propto x^5$) we would get a more precise result, but only by about 2 μm. Finally let us remark that the modern definition of parsec does not use a tangent of an angle anymore. but defines parsec explicitly as $1\ \mathrm{au} \cdot \frac{648000}{\pi}$.

35 The resulting temperature of the reflective sphere in the equilibrium state does not depend at all on what the black body inside looks like, at what temperature it would be when not covered, or how much of the radiation the outer sphere reflects back in: if the energy source supplies the power P, this energy must simply also be emitted again. The foil has two surfaces – inner and outer – and their total area is $2S$. Only half of this area radiates out, so

$$P = \frac{1}{2}\sigma 2S\widetilde{T}^4,$$

where $\widetilde{T}$ is its temperature. Before the foil was installed, in equilibrium state we had $P = \sigma S T^4$. Therefore $\widetilde{T} = T$.

36 First of all, the mirror is radially symetric so we can solve this problem only in two dimensions with two coordinates, namely z (height) and r (distance from the rotation axis). The coordinate system will be rotating with the mirror. Let us look at the forces acting on a small element of mercury. Reduced to unit mass, they are

- gravitational acceleration $\vec{a_g}$ pointing down with magnitude g;

- centrifugal acceleration $\vec{a_h}$ pointing away from the rotation axis, whose magnitude increases linearly with r as $\omega^2 r$.

Their corresponding potentials are gz and $-\frac{1}{2}\omega^2 r^2$. The total potential is therefore

$$U(r,z) = gz - \frac{1}{2}\omega^2 r^2. \tag{36.1}$$

Mercury will form a shape which has constant potential on the whole surface (if the potential wasn't constant, mercury would want to reshape); and since potential is defined up to an additive constant, we can choose it so that the potential is zero on the surface of the mercury. The shape of the surface is thus given by a line whose explicit equation we can obtain by

$$gz - \frac{1}{2}\omega^2 r^2 = 0 \quad \Rightarrow \quad z = \frac{\omega^2}{2g} r^2. \tag{36.2}$$

Here we recognise a parabola – now we only need to determine its parameters. All light rays parallel to the axis of the mirror (vertical rays in this problem) will concentrate in one point (which is a pleasant finding given that we are building a telescope). The simplest way to find the focus is to find a ray that will be horizontal after reflecting from the parabola. In the point where this particular ray hits the mirror, the mercury makes a 45° angle with the horizontal line and the z coordinate of the point is the same as that of the focus point. Or we can recall that a parabola is a set of points which are at the same distance from the focus and the directrix and that in the point where the particular ray hits the mirror we know that $z = f$ and $r = 2f$. Then

$$f = \frac{\omega^2}{2g} 4f^2 \quad \Rightarrow \quad f = \frac{g}{2\omega^2}. \tag{36.3}$$

37 At the moment when the cylinder is closed, the gas pressure is p_{atm} and the volume of the gas inside is SH, where S is the area of the cylinder's base and H is its height. After reaching equilibrium, the piston is at height h. That means that the pressure inside is increased by $\frac{kh}{S}$ by the spring acting on the piston. Since we are interested in the equilibium height h after a long time, temperature of the gas will be the same as at the beginning so we consider this as isothermic compression which means that

$$p_{\text{atm}} SH = \left(p_{\text{atm}} + \frac{kh}{S}\right) Sh. \tag{37.1}$$

The equilibrium height h is therefore[5]

$$h = -\frac{Sp_{\text{atm}}}{2k} + \sqrt{\left(\frac{Sp_{\text{atm}}}{2k}\right)^2 + \frac{SHp_{\text{atm}}}{k}}. \tag{37.2}$$

The equilibrium pressure inside is

$$p = p_{\text{atm}} + \frac{kh}{S} = \frac{p_{\text{atm}}}{2} + \sqrt{\left(\frac{p_{\text{atm}}}{2}\right)^2 + \frac{kHp_{\text{atm}}}{S}}. \tag{37.3}$$

[5] After solving the quadratic equation we utilise the positive solution.

Now let's displace the piston by x. This displacement results in decrease in pressure by Δp. We are interested in the apparent stiffness in the very first moment when the gas and the surroundings have not yet exchanged any heat. Thus we consider this to be an adiabatic process which is described by

$$p(Sh)^{\varkappa} = (p - \Delta p)[S(h+x)]^{\varkappa}. \tag{37.4}$$

Hence

$$\Delta p = p - \frac{ph^{\varkappa}}{(h+x)^{\varkappa}}. \tag{37.5}$$

The force acting on a piston is

$$F = -\Delta pS - kx = -pS\left[1 - \left(1 + \frac{x}{h}\right)^{-\varkappa}\right] - kx \tag{37.6}$$

and for small displacements

$$F = -\left(\frac{\varkappa pS}{h} + k\right)x, \tag{37.7}$$

whence we immediately see that the apparent stiffness is

$$K = -\frac{F}{x} = \frac{\varkappa pS}{h} + k = \left(\frac{2\varkappa}{\sqrt{1 + \frac{4kH}{Sp_{\mathrm{atm}}}} - 1} + \varkappa + 1\right)k. \tag{37.8}$$

For the numerical values from the problem statement the result is $K \approx 1.8\ \mathrm{kN/m}$.

38 Let's denote the charge on Paula's sticks q, their length d and mass m. We are interested in small oscillations, i. e. small angular displacement φ from the equilibrium (which is obviously the position when the stick is standing upwards) so if we encounter any formula containg φ, we can make use of the Taylor series and then neglect all terms with φ^2 and higher orders.

Three forces act upon the middle stick – gravitational and two electric – and since the stick can rotate around its bottom end, we want to know the corresponding torques. The gravitational force of magnitude mg acts in the middle of the stick so the corresponding torque at angular displacement φ is $\frac{d}{2}mg\sin\varphi \approx \frac{d}{2}mg\varphi$, where displacement φ in clockwise direction is considered positive and torque is positive if it is trying to turn the stick in clockwise direction.

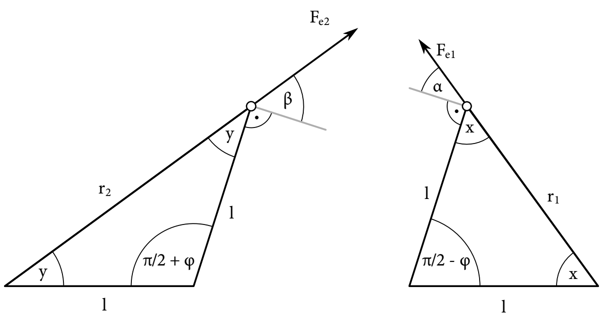


Figure 38.1: *A displaced stick and distances and angles in the triangles.*

It gets a bit complicated with moments of electrical forces. Let the distance of the upper point charge from the one on the table on the right be r_1. Applying the law of cosines to the triangle in the right half of the picture 38.1 yields

$$r_1^2 = d^2 + d^2 - 2dd\cos\left(\frac{\pi}{2} - \varphi\right) = 2d^2(1 - \sin\varphi) \approx 2d^2(1 - \varphi). \tag{38.1}$$

The square of the distance to the left charge is analogically $r_2^2 = 2d^2(1+\varphi)$. If we know the distances, we know the magnitudes of the electrical forces, but we also want to know their directions. Let's look at the picture 38.1 once again.

We have two isosceles triangles which we can use to calculate the angles

$$x = \frac{\pi}{4} + \frac{\varphi}{2} \quad \text{a} \quad y = \frac{\pi}{4} - \frac{\varphi}{2}, \tag{38.2}$$

because the sum of interior angles in a triangle is π. Furthermore, we also see that

$$\alpha + \frac{\pi}{2} + x = \pi \quad \text{a} \quad \beta + \frac{\pi}{2} + y = \pi, \tag{38.3}$$

so

$$\alpha = \frac{\pi}{4} - \frac{\varphi}{2} \quad \text{a} \quad \beta = \frac{\pi}{4} + \frac{\varphi}{2}. \tag{38.4}$$

To calculate the torque, we need to know cosines of these angles, so

$$\cos\alpha = \cos\frac{\pi}{4}\cos\frac{\varphi}{2} + \sin\frac{\pi}{4}\sin\frac{\varphi}{2} \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{\varphi}{2} \tag{38.5}$$

and analogically

$$\cos\beta = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\frac{\varphi}{2}. \tag{38.6}$$

Finally we are getting to the torques where we use Taylor series $\frac{1}{1-\varphi} \approx 1+\varphi$ and $\frac{1}{1+\varphi} \approx 1-\varphi$. Angular acceleration ε is then calculated as

$$\begin{aligned} J\varepsilon &= -d\frac{q^2}{4\pi\varepsilon_0 r_1^2}\cos\alpha + d\frac{q^2}{4\pi\varepsilon_0 r_2^2}\cos\beta + \frac{d}{2}mg\sin\varphi \\ &\approx -d\frac{q^2}{4\pi\varepsilon_0 2d^2}(1+\varphi)\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{\varphi}{2}\right) + d\frac{q^2}{4\pi\varepsilon_0 2d^2}(1-\varphi)\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\frac{\varphi}{2}\right) + \frac{d}{2}mg\varphi \\ &\approx -\left(\frac{q^2}{4\pi\varepsilon_0}\frac{3\sqrt{2}}{4d} - \frac{d}{2}mg\right)\varphi. \end{aligned} \tag{38.7}$$

The moment of inertia of a stick rotating around its end is $J = \frac{1}{3}md^2$, which, after plugging in into previous equation, yields

$$\varepsilon = -\frac{3}{md^2}\left(\frac{q^2}{4\pi\varepsilon_0}\frac{3\sqrt{2}}{4d} - \frac{d}{2}mg\right)\varphi, \tag{38.8}$$

which is a simple harmonic oscillator with angular frequency

$$\omega^2 = \frac{3}{md^2}\left(\frac{q^2}{4\pi\varepsilon_0}\frac{3\sqrt{2}}{4d} - \frac{d}{2}mg\right). \tag{38.9}$$

Thus, the period of small oscillations with the numerical values from the problem statement is

$$T = \frac{2\pi}{\omega} \doteq 1.95\ \text{s}. \tag{38.10}$$

$\boxed{39}$ The light bulb shines equally in all directions, so we are able to express its luminous intensity I as a fraction of its total flux Φ,

$$I = \frac{\Phi}{4\pi}. \tag{39.1}$$

Apart from this we will also need to know its *luminous emittance*, or the luminous flux per unit surface area when looking along any ray ending on its surface. If we denote its radius r, the area of its cross-section is πr^2 and emittance

$$L = \frac{I}{\pi r^2} = \frac{\Phi}{4\pi^2 r^2}. \tag{39.2}$$

How does it look under the lamp? The reflector redirects a fraction of the light onto the pavement, increasing the illumination in comparison to a naked bulb. Since the bulb is exactly in the focus of the paraboloid, the intensity will increase markedly along its axis. The bulb is not a point source, so these rays will not be perfectly parallel – however if we are looking from a place sufficiently far below the lamp, where the angles in question are already very small, we can look in any direction and see

- either the sky or the lamppost (which are of no interest to us since they do not emit any light);
- or the bulb (and therefore emittance L);
- or its reflection from the polished surface of the reflector (and again emittance L).

For this ratio of sizes of the bulb and the mirror and the height of the lamp this condition certainly holds. On the whole, when looking from below the lamp behaves as if it had a larger bulb with the same diameter as the reflector, but with same emittance as the real bulb[6] Now we only need to find the illuminance, which can be done by finding the solid angle which the reflector subtends when looking from the surface of the road, and multiply it by luminance. Since all angles are small, we can approximate the solid angle as

$$\Omega \approx \pi\left(\frac{R}{H}\right)^2, \tag{39.3}$$

where R is the radius of the reflector and $H \gg R$ the height of the lamppost. The illuminance under the lamp will then be

$$E = L_0\Omega \approx \frac{\Phi}{4\pi^2 r^2}\pi\left(\frac{R}{H}\right)^2 \approx 354\ \text{lx}. \tag{39.4}$$

[6] The total luminous flux is of course conserved: only light that would have been emitted upwards or to the sides, is now reflected down. Only light that reflects back onto the bulb cannot exit the reflector.

40 Kepler's laws tell us that if we throw an object in radial gravitational field of the Moon, it will follow an elliptical trajectory with one of its foci F_1 in the center of the Moon. We only need to find the position of the second focus point so that the initial speed will be minimal. Let us denote the angular distance between Matthew and Jacob as φ (in our case, $\varphi = 90°$, because we are throwing the rock from the North Pole to the equator). It is clear that the other focus point will lie on the bisector of this angle (see figure 40.1). To find this point we can use the *vis-viva equation*.

It relates the total energy of an object orbiting a planet with the semi-major axis of the ellipse. We will use it in form

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a} \tag{40.1}$$

where the expression on the left-hand side is the total energy of the orbiting body (kinetic and potential) and a is the semi-major axis of its orbit.

The derivation of this equation relies on using the conservation laws for energy and angular momentum. Consider a point mass orbiting a planet of mass M and let us denote its speed at the pericentre as v_p and its distance from the planet as r_p; and at the apocentre v_a and the distance r_a respectively. The energy conservation law tells us that

$$\frac{1}{2}mv_p^2 - G\frac{Mm}{r_p} = \frac{1}{2}mv_a^2 - G\frac{Mm}{r_a} \tag{40.2}$$

and the angular momentum conservation law yields

$$mv_pr_p = mv_ar_a \quad \Rightarrow \quad v_a = v_p\frac{r_p}{r_a}. \tag{40.3}$$

If we plug this into the equation 40.2, we get

$$\frac{1}{2}mv_p^2 - G\frac{Mm}{r_p} = \frac{1}{2}mv_p^2\frac{r_p^2}{r_a^2} - G\frac{Mm}{r_a} \quad \Rightarrow \quad \frac{1}{2}mv_p^2\left(1 - \frac{r_p^2}{r_a^2}\right) = GMm\left(\frac{1}{r_p} - \frac{1}{r_a}\right). \tag{40.4}$$

Finally, dividing by the expression $\left(1 - \left(\frac{r_p}{r_a}\right)^2\right)$ and using $2a = r_a + r_p$ gives us the result

$$\begin{aligned} \frac{1}{2}mv_p^2 &= \frac{GMm}{2a}\frac{r_a}{r_p} \\ \frac{1}{2}mv_p^2 - G\frac{Mm}{r_p} &= -\frac{GMm}{2a}. \end{aligned} \tag{40.5}$$

The law of conservation of energy says that the expression $mv^2/2 - GMm/r$ is constant along the object's trajectory. We have managed to express it at the point of its closest approach, therefore it must be the same for any other point on the trajectory. Thus the vis-viva equation (40.1) is proven.

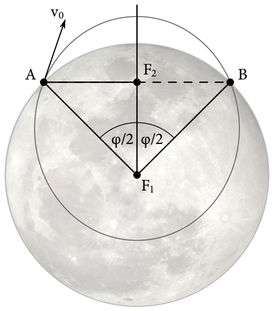


Figure 40.1: *The trajectory of the rock.*

Let us denote the unknown initial speed of the rock as v_0. Then the equation 40.1 becomes

$$\frac{1}{2}mv_0^2 - G\frac{Mm}{R} = -G\frac{Mm}{2a} \quad \Rightarrow \quad v_0^2 = GM\left(\frac{2}{R} - \frac{1}{a}\right). \tag{40.6}$$

Notice that the initial speed v_0 decreases with decreasing semi-major axis. Therefore we only need to find the position F_2 of the second focus so that the length of the semi-major axis is minimal. Next we notice that if we take any point on the ellipse, e. g. point A, and we calculate the sum of the distances from this point to the foci of the ellipse ($|AF_1| + |AF_2|$ in the figure), we always get $2a$. However, the distance $|AF_1|$ is always equal to the radius of the Moon $R_☾$, so we can only minimize $|AF_2|$.

The shortest possible distance corresponds to the situation when point F_2 lies on the line connecting A and B. Then

$$2a = |AF_1| + |AF_2| = R_☾\left(1 + \sin\frac{\varphi}{2}\right) \tag{40.7}$$

and if we plug this into equation 40.6 and express the speed v_0, we get

$$v_0 = \sqrt{\frac{2GM_☾}{R_☾}\left(1 - \frac{1}{1 + \sin\frac{\varphi}{2}}\right)}. \tag{40.8}$$

Setting $\varphi = 90°$ yields

$$v_0 = \sqrt{\frac{GM_☾}{R_☾}\frac{2\sqrt{2}}{2 + \sqrt{2}}}. \tag{40.9}$$

Answers

1 0.17 *exactly*.

2 01:10:00

3 145 m

4 60 °C

5 $\frac{11}{15}$

6 270 m

7 5 Ω and 20 Ω, *in any order*.

8 $h = \frac{2k^2}{g}$

9 $\frac{3}{7}$

10 90°

11 $\left(\frac{3a-b}{12}\right)b^2 \tan\beta$

12 36 km/h

13 6

14 391 m^3

15 $\frac{4}{9} + \frac{26\sqrt{2}}{9\pi} = \frac{4\pi + 26\sqrt{2}}{9\pi}$

16 12 mm

17 $\mathrm{d}W = m\omega^2 \ell \,\mathrm{d}\ell$

18 $2\frac{1+\sqrt{13}}{13+2\sqrt{13}} = 2\frac{11\sqrt{13}-13}{117} \doteq 46\ \%$

19 126.75 m

20 $\frac{5}{3}a$

21 1.56 m

22 68°

23 0.13 °C

24 $5g$

25 170 cm

26 0.5 m/s

27 The full hippo is $\frac{\sqrt{202}}{14}$ times faster.

28 392 m. *Accept results in range 385 – 405 m.*

29 $\frac{3\pi r^4}{3\pi r^4 + a^4}\omega$

30 $\frac{T_0 + T_1}{2}$

31 md^2

32 $\frac{2}{\sqrt{3}}\frac{mg}{L}$

33 $\frac{\left(2+\frac{\pi}{2}\right)}{\left(17+\frac{\pi}{4}\right)} \doteq 0.2 = 20\ \%$

34 364 km

35 T

36 $\frac{g}{2\omega^2}$

37 1.8 kN/m

38 1.95 s

39 354 lx

40 $\sqrt{\frac{GM}{R}\frac{2\sqrt{2}}{2+\sqrt{2}}}$