

# Feladatok

**1** Jusstínia nagyí a Geochimica et Cosmochimica Acta folyóirat mellékletében egy grafilógikai feladványt talált. Miután sikerült megoldania, úgy döntött, megkeresi a kép súlypontját. Milyen messze lesz ez a kép geometriai középpontjától, ha minden beszínezett négyzetnek megegyező tömege van, és az üres négyzeteknek nincs tömegük? A négyzetek oldalhossza 1 egység.

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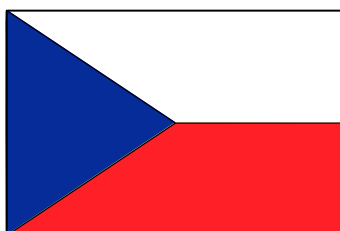
**2** Márk harminc perccel a kezdete után csatlakozott egy élő videóközvetítéshez. A visszajátszási sebességet  $1,75\times$  értékre állította. Mennyi időbe fog telni neki, hogy utolérje az élő közvetítést?

A választ óó:pp:mm formátumban küldje be!

**3** Egy túrázó éppen felért egy magas hegy csúcsára. A teljes magasságkülönbség a túrája során 1302 méter, míg a mászás során felfelé megtett út 1447 méter volt. Hány métert kell felfelé másznia a visszaút során, ha ugyanazt az ösvényt választja, mint amin a csúcs felé haladt?

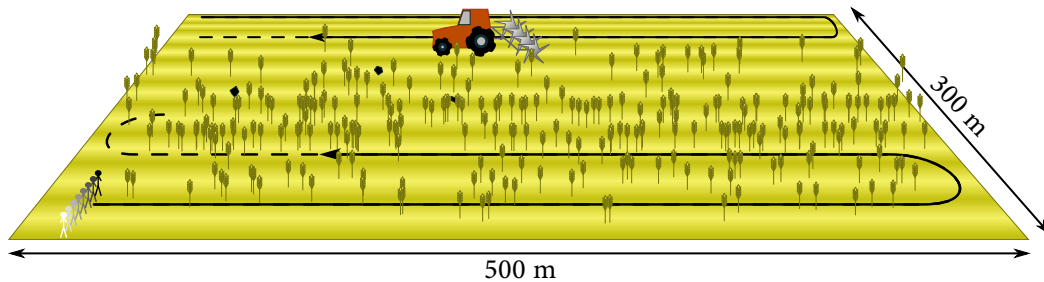
**4** Egy közelmúltban tartott ünnep tiszteletére összeállítottuk a 54 cm széles és 36 cm magas csehszlovák zászlót három darab színes fémlemezéből. A kék háromszög a zászló teljes szélességének a feléig terjedt. Amikor szabadtéren volt kiállítva, a zászlót megvilágította a Nap, és minden része különböző hőmérsékletűre melegedett: a fehér rész  $30\text{ }^\circ\text{C}$ -ra, a piros  $70\text{ }^\circ\text{C}$ -ra, a kék pedig  $90\text{ }^\circ\text{C}$ -ra melegedett fel.

Mekkora lesz a zászló hőmérséklete, ha ezután elszigeteljük a környezetétől, és megvárjuk a hőmérsékleti egyensúly beálltát?



5 Egy nyolcfős csapat egy nemrég lehullott meteorit után kutat egy mezőn. A csapat tagjai zárt láncban haladnak. Mindenki egy 2,5 m széles csíkot figyel, miközben 1 m/s sebességgel sétál. Valahányszor a mező szélére érnek, gyorsan megfordulnak, átmennek szomszédos földszávrá, és ott folytatják a keresést. A mező 500 m hosszú és 300 m széles.

Azonban amikor a csapat megkezdte a keresést, egy traktor 300 m távolságban elkezdte szántani a mezőt. A traktor ekéje 10 m széles és 18 km/h sebességgel halad. A mező szélére érve ez is nagyon gyorsan megfordul. Mekkora része marad a mezőnek **átkutatlanul**, ha a kutatást azonnal abba kell hagyni, amint a csapat **bármelyik** tagja eléri a traktort?



6 Ismét a kényelem kerekedett felül a környezettudatosságon... Marcel, Kevin, Patrik és Jakab a vasútállomás felé hajtanak, mindenki a saját autójában. A faluban a maximális megengedett sebességgel, 50 km/h-val haladnak, eközben 150 m hosszú részt foglalnak el az úton. Ha mindenki ugyanúgy vezet, mekkora területet foglalnak majd el a falun kívül az úton, miután 90 km/h sebességre gyorsultak?

*Az autók hosszát elhanyagolhatjuk.*

7 Dani véletlenszerűen kiválasztott két ellenállást a barkácsdobozából és bekötötte őket egy áramkörbe, először soros, majd párhuzamos kapcsolásban. Amikor este a takarítónő betévedt a rejtékébe, többek között egy cetlit talált az asztalán rajta két értékkel:  $4 \Omega$  és  $25 \Omega$ .

Mekkorák voltak az egyes ellenállások értékei?

8 Egy középkori kastélyban egy idegenvezető bemutatja a híres kútát a kíváncsi látogatóknak. Az idegenvezető, alábecsülve a látogatók matematikai képességeit, egy *egyszerűsített* módszert ismertet a mélység kiszámítására: a látogató dobjon egy érmét vagy egy követ a kútba, és számolja az időt, amíg meghallja az érme csattanásának hangját a fenéken. Ezután a mélység megállapításához ezt az időt szorozzák meg valamilyen állandó  $k$  értékkel. Természetesen az idegenvezető úgy határozta meg a  $k$  konstans értékét, hogy ez a módszer valóban helyes értéket eredményezzen az adott kútnál. Fejezd ki a kút mélységét a  $k$  állandóból és a  $g$  gravitációs gyorsulásból.

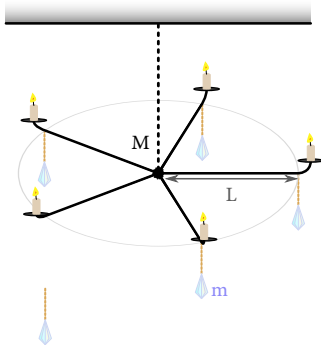
*A hangsebesség minden középkori kútban végtelen nagy.*

9 Juli acetont vízben oldott. Úgy alakult, hogy el kellett hagynia a laborot egy pillanatra. Amikor visszajött, azt tapasztalta, hogy az acetontömegének egyharmada és a víz tömegének egytizede elpárolgott. Juli megmérte, hogy az acetontömegaránya az oldatban egyhatodával csökkent.

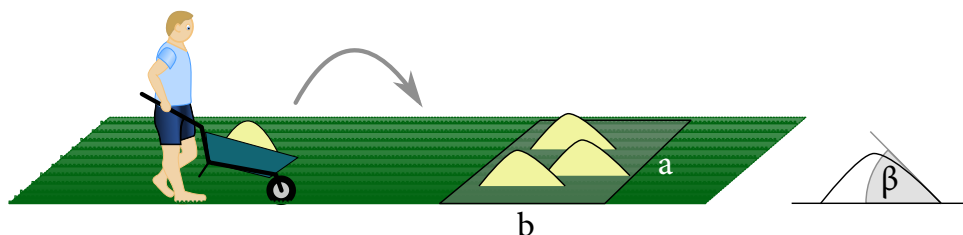
Mekkora volt az acetontömegaránya az oldatban?

**10** Vikinek van egy csillárja, amely egy pontszerűnek tekinthető,  $M$  tömegű középpontból, és abból egy síkban kiálló öt darab, elhanyagolható tömegű,  $L$  hosszúságú rúdból áll, amelyek egyenlő szöget zárnak be egymással. Mindegyik rúd végén egy  $m$  tömegű gyertya található. A teljes csillár egy, a pontszerű középponthoz kapcsolódó, hosszú lánccal van a plafonhoz rögzítve.

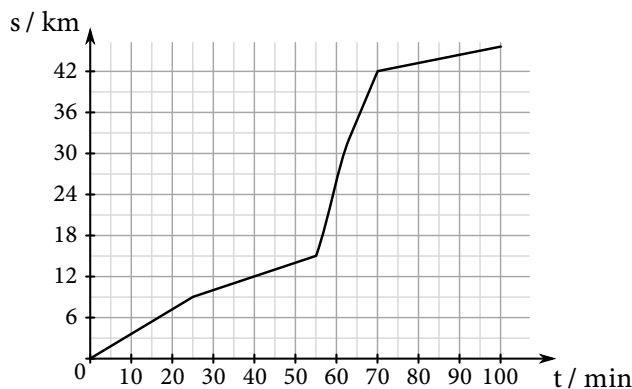
Hirtelen leesik az egyik gyertya. Miután a rendszer egyensúlyba került, mekkora szöget zár be vízszintessel a rudak által meghatározott sík?



**11** Vilmos vásárolt egy téglalap alakú telket  $a$  és  $b$  oldalakkal, ahol  $a \geq b$ . Mennyi homokot tud odahordani úgy, hogy az a telken belül maradjon? A homok lejtőszöge  $\beta$ .



**12** Jakab kipróbált egy új, tömegközlekedést használók számára fejlesztett alkalmazást. Ez többek között rögzíti a megtett távolságot, illetve kijelzi az indulástól számított átlagsebességet is km/h -ban. Jakab az előző estét buszozással töltötte, ma pedig grafikonon ábrázolta az általa megtett távolságot az idő függvényében az alább látható módon. Mi volt a legnagyobb átlagsebesség, amit az alkalmazás mutatott Jakab útja alatt?



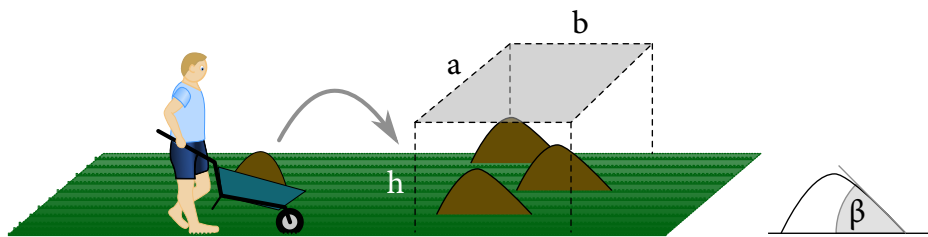
**13** Teréz vásárolt egy különleges teáskészletet Angliában. A készlet egy közepesen kiegyensúlyozott mérlegből áll és öt csészélejből, amelyek a mérlegen egymástól egyenlő távolságra helyezkednek el. Teréz áthívja a

barátait teázni. Az öt barát ABC sorrendben ül le a teáskészlethez. Teréz teával kínálja őket, így a csészealjakra helyez öt teáscsészét, amelyek tömegeinek aránya  $1 : 2 : 3 : 4 : 5$ .

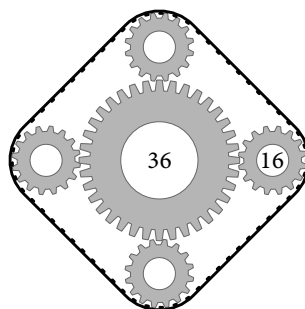
Hány különböző módon oszthatja ki barátainak a teáscsészéket Teréz, ha a mérlegnek egyensúlyban kell maradnia?



**14** Vilmos szeretne egy szobrot is építeni a telkén. Az emlékmű alapja egy téglalap alakú betonlap lesz  $a = 10$  m és  $b = 7$  m oldalakkal. Az alapot  $h = 3$  m magasságban kell elhelyezni a felszín felett és teljes területe alatt alá kell támasztani. Minimum mekkora térfogatú földkupacot kell Vilmosnak a betonlap alá hordania, ha a föld lejtőszöge  $\beta = 45^\circ$ ?

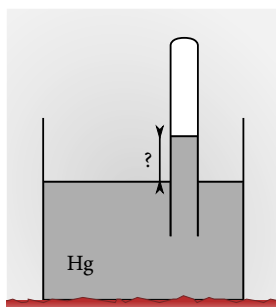


**15** A születésnapjára András egy építőkészletet kapott. Azonnal épített is egy különleges gépet: egy 36 fogból álló fogaskerék köré 4 db kisebb, 16 fogból álló fogaskereket helyezett. Ezután az egész szerkezet köré egy szíjat erősített. Hányszor kell a nagy fogaskereket megforgatnia, hogy a szíj egyszer körbeforduljon? Az összes fogaskerék tengelye rögzített.

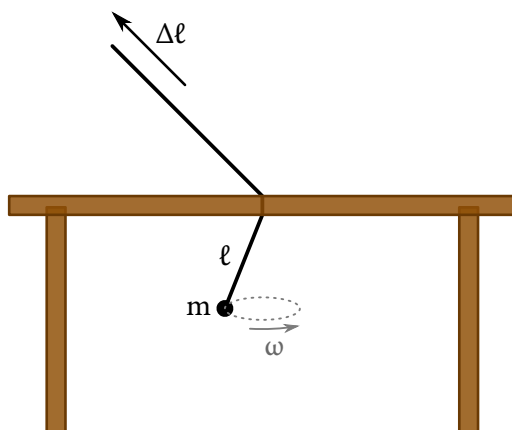


**16** Tamás asztronautaként elment egy marsi expedícióra, ahová magával vitte az örökségül kapott barométerét, amiben egy higanyoszlop található. Az oszlop 760 mm magas volt a Földön. Mekkora az oszlop magassága a Marson, ahol a felszíni légköri nyomás 600 Pa?

*Tekintsünk el mindkét bolygó forgásától.*

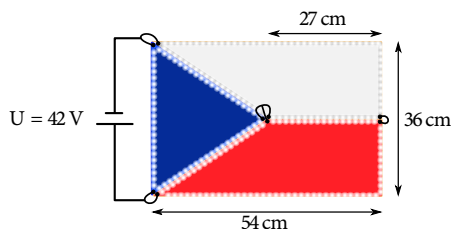


- 17** Vilma áthúzott egy fonalat az asztalon levő lyukon keresztül úgy, hogy a fonál asztal alatti részének hossza  $\ell$ . Ezután felakasztott rá egy  $m$  tömeget és megpörgette egy körpályán  $\omega$  szögsebességgel. Természetesen a fonál felső végét a kezében tartotta. Mekkora munkát kell végeznie, hogy a fonalat egy kis  $\Delta\ell$  távolsággal maga felé húzza?



- 18** Egy fém zászló sokaknak megdobogtatja a szívét, azonban sötétben nemigen lehet látni. Ezért fogtunk egy 54 cm széles és 36 cm hosszú csehszlovák zászlót és 30  $\Omega/\text{m}$  fajlagos ellenállású fényszalagokkal világítottuk meg. Minden színt közrezárt egy-egy szalag minden oldalról, és egyes tartományok határos csúcsainál összekötöttük a szalagokat. Ezután a zászló két baloldali sarkára 42 V bemeneti feszültséget kapcsoltunk.

A biztonsági technikusunknak azonban voltak fenntartásai az áramkörünk munkabiztonságát illetően, ezért lekapcsolta a feszültségforrást, kiválasztott véletlenszerűen egy pontot a szalagon és beillesztett egy biztosítékot 2 A névleges áramkorláttal. Mi a valószínűsége annak, hogy a biztosíték leolvad, amikor a bemeneti feszültséget újra bekapcsoljuk?

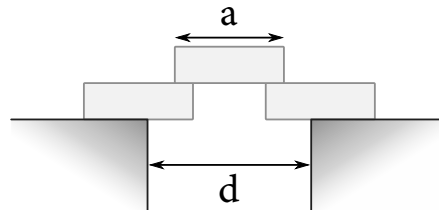


- 19** Lujza, a mozdonyvezető meghúzta a vészféket a vonaton. A fékek pneumatikusan működnek és enyhe késleltetéssel fejtik ki a hatásukat: a mozdony kerekei azonnal, az első kocsié egy másodperccel később, majd minden ezután következő kocsi kerekei 1 s -cel az előző kocsi kerekei után blokkolódnak.

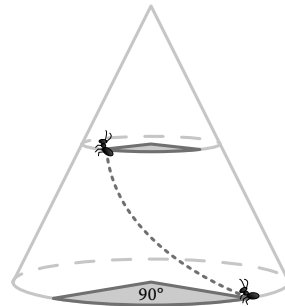
A vonatnak öt kocsija van, mindegyik 20 t, a mozdony súlya 100 t, a kerekek és a sínek közötti csúszási súrlódási együttható pedig 0,2. Mekkora a teljes féktáv, ha a szerelvény fékezés előtti sebessége  $\{72 \text{ km/h}$ ?

**20** Ádám szeret hidakat építeni. Ezúttal három olyan téglatest alakú építőelemet választott ehhez, amelyeknek élei  $a \geq b \geq c$  hosszúságúak. Az egyes elemek közötti, valamint az elemek és a talaj közötti súrlódás kicsi, ezért Ádám nem tud erre hagyatkozni az építéskor. Mekkora az a legnagyobb távolság, amelyet át tud hidalni csak ezen három elem felhasználásával?

*Az építőelemek elülső oldalai egy síkban kell legyenek.*

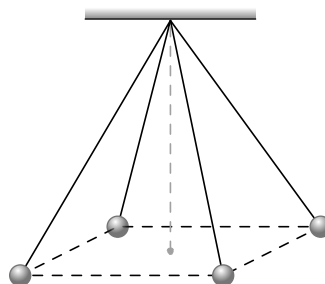


**21** Egy hangya egy 1 m sugarú és 2 m magasságú, tölcser alakú süvegcsukorra készül felmászni. Jelen pillanatban a cukortölcser alapjánál álldogál, pontosan délkeletre a csúcstól. Mi a legrövidebb távolság, amit másznia kell ahhoz, hogy a tölcser délnyugati oldalára jusson 1 m magasságban?



**22** Paula talált otthon négy egyforma, homogén, 2 kg tömegű és 2 m hosszú rudat. Lelógatta őket a plafon egy pontjáról, majd egy-egy  $10^{-4} \text{ C}$  nagyságú, tömeg nélküli ponttöltést rögzített mindegyik rúd végére, aminek hatására azok taszítani kezdték egymást. Határozza meg a pálcák függőlegessel bezárt szöget az egyensúlyi állapotban!

Ennek a problémának nincs zárt-alakú megoldása. Nyugodtan használjon számológépet. Válaszát fokban adja meg, legalább egy tizedesjegy pontossággal.

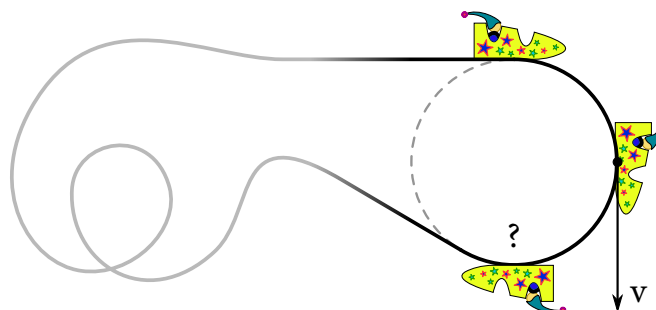


**23** Egy teherautó 50 km/h sebességgel halad az úton. Egy légy repül vele szembe ugyanekkor a sebességgel. Az ütközés pont úgy zajlik, ahogy várható; a légy pépként szétkenődik a szélvédőn. Mennyivel nő meg a pép hőmérséklete, ha a légy fajlagos hőkapacitása  $3 \text{ kJ}/(\text{kg}\cdot\text{K})$ ?

*Tételezzétek fel, hogy a szélvédő tökéletesen rugalmatlan, és nem melegszik fel.*

**24** Máté a sikeres záróvizsgálata után egy vidámparkban talált munkát. A legkedvesebb mutatványa a hullámvasút egy függőleges félhurokkal. Máté a hurok tetején beleül egy kocsiba, és gyengén ellöki magát a korláton, hogy elinduljon. Amikor félúton van lefelé, a sebessége  $v$ . Mekkora nehézségi gyorsulást fog Máté érezni (a gravitációs gyorsulás hányszorosát), amikor a félhurok legalsó pontján halad át a kocsival?

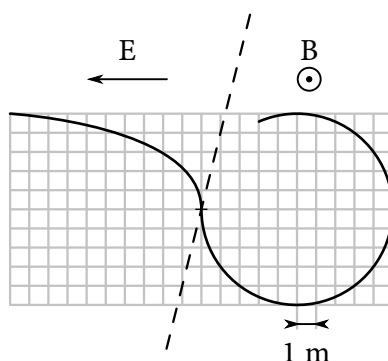
*A légellenállás, a súrlódás a kocsi és a pálya között és hasonló csúnya erők elhanyagolhatóak.*



**25** Jancsi szobájában a plafonról egyforma rugók lógnak, melyek nyugalmi hosszúsága ismeretlen. A rugók közel vannak egymáshoz. Amikor Jancsi egy rugóba kapaszkodva lóg, 50 cm magasan van a padló felett. Amikor két rugóba kapaszkodik, 140 cm magasan van a padló felett.

Milyen magasan lesz Jancsi a padló felett, ha három rugóba kapaszkodva lóg? A plafon Jancsi szobájában 250 cm magasan van.

**26** Patrik talált egy különös elektromosan töltött golyót. Hogy többet megtudjon róla, egy modern elektromágneses analitikai kamra felhasználásával elemezte tulajdonságait. A kamrában eltekinthetünk a gravitációtól. A berendezés két részre van felosztva: a bal oldalán homogén,  $E = 0,8 \text{ V/m}$  nagyságú elektromos tér, a jobb oldalán pedig homogén,  $B = 0,4 \text{ T}$  nagyságú mágneses tér található. A különös golyót behelyezte a kamrába, a kirajzolódó pályáját a mellékelt ábra mutatja. Mi volt a golyó kezdeti sebessége?



**27** A Nílus  $\alpha$  meredekségű partján,  $h$  magasan a vízszint felett napozik két homogén, gömb alakú víziló. Mindkét víziló gömb alakú gyomra a teste közepén helyezkedik el. Az egyikük még éhes, a másikuk viszont már jóllakott a fűből, vízből és elveszett turistákból álló étkével, amik sűrűsége megegyezik a vízilóéval. Az éhes víziló tömege egy nyolcaddal kevesebb, mint a jóllakotté.

Amikor már eleget napoztak, elkezdenek visszagurulni a vízbe. A vízbeéréskor melyik lesz gyorsabb, és mennyiszor?

*Egy víziló gyomrának tartalma együtt forog a víziló többi részével, valamint a súrlódás a Nílus partján elég nagy, hogy a vízilóvak ne csússzanak meg.*

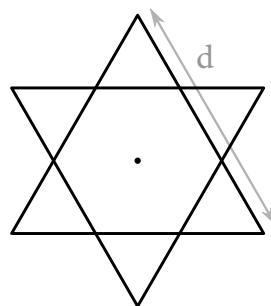
**28** Lujza levegővel felfújta egy 100 g tömegű és 10 cm sugarú strandlabdát légköri nyomására. Ezt követően elkezdte lassan víz alá nyomni. Mekkora az a legnagyobb lehetséges mélység, ahonnan a labda külső hatás nélkül még feljön a víz felszínére, feltéve, hogy a víz hőmérséklete mindenhol állandó?

**29** Kivágtunk egy  $r$  sugarú, kör alakú, és egy  $a$  oldalhosszú, négyzet alakú lapot egy vékony,  $\sigma$  felületi sűrűségű fémlapból. Ezután a két lapot a középpontjuk mentén, merőlegesen felfűztük egy rúdra. A körlapot elkezdjük forgatni  $\omega$  körfrekvenciával, majd hozzáértettük a négyzet alakúhoz. Mi lesz a végső közös körfrekvenciája a testeknek azután, hogy a súrlódás miatt beáll egy közös forgássebesség?

**30** Dániel azt vette észre, hogy ha egy ellenálláson áram folyik át, az felmelegszik. A kedvenc ellenállása, amit egy állandó feszültségű telephez kötött, szintén felmelegedett – hosszú idő után a hőmérséklete  $T_1$ -en stabilizálódott. Ezután a kísérlet után megismételte mindezt egy másik ellenállással is. Ez az ellenállás ugyanabból az anyagból készült, de minden dimenziója kétszer akkora volt. Mi a kialakuló stabil hőmérséklete a második ellenállásnak?

Mindkét esetben egy nagy ventilátort irányítottunk az ellenállásokra, ami elfújta a meleg levegőt róluk, és  $T_0$  hőmérsékleten tartotta a környezetüket. A számolásban a hősugárzástól tekintünk el.

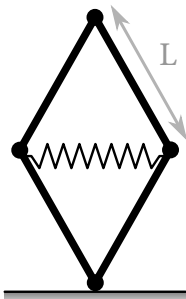
**31** Egy Pozsony utcáin tett esti séta során Szofi észrevette, hogy a karácsonyi dekorációk már fel vannak szerelve. Először elcsodálkozott azon, hogy már megint novemberben kezdődik a Karácsony, de aztán észrevett egy érdekes, hatágú csillag alakú dekorációt. Hat rudacsából készült, amelyek egyenként  $d$  hosszúak és  $m$  tömegűek. Igazi fizikusként azonnal kiszámolta a dekoráció tehetetlenségi nyomatékát a csillag síkjára merőleges, a középponton áthaladó tengelyhez mérve. Milyen eredményt kapott?



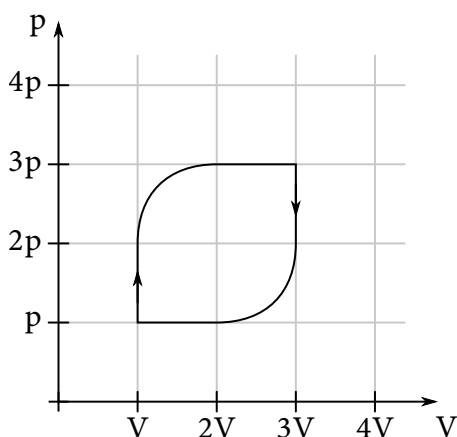
**32** Dávid az ígéretes fizikusi karrierje mellett a modern művészettel is foglalkozik. Felrohan a padlásra, és lehozott onnan négy botot, amelyek tömege egyenként  $m$ , hossza  $L$ , és egy rugót, melynek nyugalmi hossza



nulla. Ezután úgy alkotott négyszöget a botokból, hogy azok szabadon foroghassanak a csatlakozási pontok körül, miközben végig egy síkban maradnak. Összekapcsolt két szemközti sarokpontot a rugóval, és úgy állította egy sarokpontjára a művét, hogy a rugó vízszintes helyzetben legyen. Mekkora a rugó rugóállandója, ha a fent leírt elhelyezésben  $L$  hosszúságúra nyúlik?

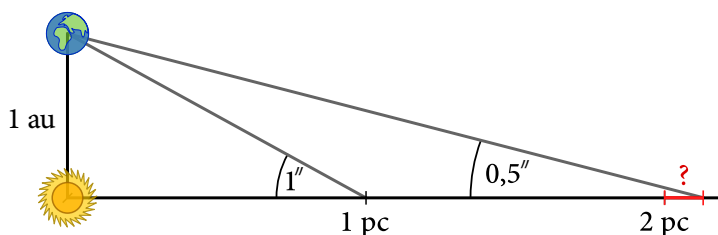


**33** Az energiaválság közepette Jaro előhúzta üknagyapja gőzgépét az istállóból. Néhány apróbb javítást követően jobb volt, mint új korában. A masina most a  $pV$  diagramon látható ciklusnak megfelelően működik. Mi a gép hatásfoka, ha a gázt ideális, egyatomos gázként kezelhetjük?



**34** Egy parszek úgy volt meghatározva, mint a távolság, amelyből a Föld keringési sugara egy szögmásodperc ( $1''$ ) alatt látszik. Martin megpróbálja kiszámítani egy  $2$  pc-re lévő csillag távolságát úgy, mintha a földpálya sugara onnan nézve  $0,5''$  alatt látszana. Martin közben rájön, hogy ez nem teljesen helyes, de mivel ezek a szögek nagyon kicsik, tudja, hogy a relatív hiba elhanyagolható lesz. Mégis szeretné tudni: hány kilométerrel különbözik az így kapott távolság az igazi  $2$  pc távolságtól?

*Ha a számológépek nem akar együttműködni lenni, akkor érdemes lehet megpróbálni a Taylor-sor használatát.*



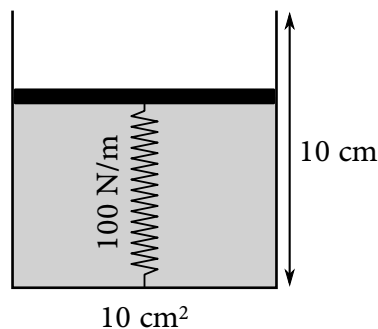
**35** Egy tökéletesen fekete, gömb alakú bolygó saját  $P$  teljesítményétől melegszik. Hőmérséklete külső hatás nélkül  $T$  értéken stabilizálódik. Most beborítjuk egy, a bolygóval megegyező sugarú, vékony, gömb alakú, fényvisszaverő réteggel. A réteg a beeső sugárzás 80 % -át visszaveri a bolygóra, míg a többit elnyeli. Mekkora a réteg egyensúlyi hőmérséklete, ha nem érinti közvetlenül a bolygó felszínét?

**36** Tamás egy hatalmas távcső megépítését tervezi. Mivel kissé aggódott, hogy a drága főtükör könnyen eltörhet, ezért úgy döntött, hogy üveg helyett folyékony higanyt használ: a higanyt egy nagy lapos tálba öntötte, és a tálát  $\omega$  szögsebességgel megpörgette.

Mekkora a tükör fókusztávolsága, ha az homogén,  $g$  nehézségi gyorsulású gravitációs mezőben van elhelyezve?

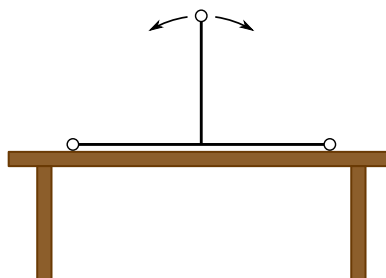
**37** Marci bezárt egy  $100 \text{ N/m}$  rugóállandójú és nulla nyugalmi hosszúságú rugót egy  $10 \text{ cm}^2$  alapterületű és  $10 \text{ m}$  magasságú hengerbe. A henger ideális kétatomos gázzal van töltve. Marci hozzáerősítette a rugó egyik végét a henger aljához és a másik végét egy nulla tömegű dugattyúhoz, ami hermetikusan lezárja a hengert. Amikor lezárta a hengert a dugattyúval, standard légköri nyomás volt benne, ami nyilván megnövekedett, mire kialakult az egyensúlyi hőmérséklet és dugattyúmagasság.

Mi lesz a látszólagos rugóállandója a bezárt rugónak kis kitérésekre az egyensúlytól?



**38** Paula ismét mechanikai rendszereket állít össze. Ezúttal három, homogén rúdja van, melyek tömege  $2 \text{ kg}$ , hossza  $2 \text{ m}$ , és mindegyiknek található a végén egy tömeg nélküli,  $10^{-4} \text{ C}$  nagyságú ponttöltés. A háromból két rudat egymással párhuzamosan a vízszintes asztallaphoz ragasztott úgy, hogy a töltéssel nem rendelkező végük összeért.

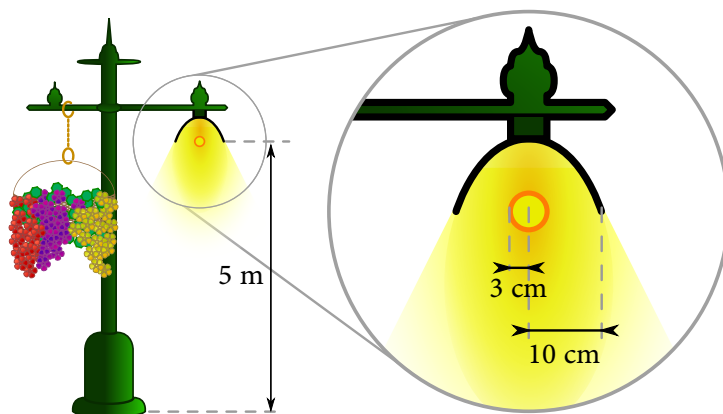
Ezt követően fogta a harmadik rudat és függőleges pozícióban rögzítette egy zsanérral abban a pontban, ahol az előző két rúd összeért úgy, hogy a töltött vége legyen felül. A harmadik rudat a zsanér csak a három elem által kifeszített, függőleges síkban engedte szabadon mozogni. Ezután enyhén kimozdította a harmadik rudat a függőleges pozíciójából, és megfigyelte annak apró rezgéseit. Mi volt a rezgés periódusideje?



**39** Egyetlen utcai lámpa áll a móló szélén, amelynek különleges, paraboloid alakú fényvisszaverő lámpabúrája van. A búra nyílásának sugara 10 cm, mélysége 7 cm és belső felülete tökéletesen veri vissza a fényt.

A fókuszában egy gömb alakú villanykörte található, amelynek sugara 3 cm, és izotróp módon (minden irányba egyenlően) sugároz ki 10 klm-nyi teljes fényáramot. Mekkora az út éppen a lámpa alatt lévő pontjának a megvilágítása luxban mérve?

*Kis szögekre használható az összefüggés, miszerint  $\sin x \approx \tan x \approx x$ . A villanykörte tökéletesen átlátszatlan.*



**40** Máté és Jakab felfedezi a Hold felszínét. Máté az északi sarkon landolt, míg Jakab továbbment az egyenlítőhöz. Rövid idő múlva Máté talált egy nagyon érdekes követ, amit el akart küldeni Jakabnak analízisre. Ehhez tudni szeretné, mekkora minimális sebességgel kell eldobnia a követ, hogy az elérje a Hold egyenlítőjét. A Hold tömege  $M_{\zeta}$  és sugara  $R_{\zeta}$ .

# Megoldások

**1** To find the centre of mass we need not solve the nonogram at all: we only need to find the distribution of mass, which is completely described by the numbers in the legend. Of course, we *may* solve it, but it costs time, which is at premium during the competition.

We should immediately notice that the nonogram is symmetric around the vertical axis, and that the horizontal coordinate of the centre of mass is the same as that of the centre of the picture. Hence, we only need to calculate the vertical coordinate of the centre of mass using the numbers in the rows. We assign each row its weight  $-6, -5, \dots, 5, 6$  corresponding to its distance from the middle row. Then we sum all the numbers in each row, multiply the sum by the row's weight and then sum it all up.<sup>1</sup> Finally, we divide it by the number of all coloured squares in the nonogram, which conveniently happens to be exactly 100.

Thus we obtain the vertical coordinate of the centre of mass

$$y = \frac{\sum_{j=-6}^6 \left( j \cdot \sum_i x_{ij} \right)}{\sum_{j=-6}^6 \left( \sum_i x_{ij} \right)} = 0,5 = \frac{(-6) \cdot 9 + (-5) \cdot (2 + 5 + 2) + \dots + 5 \cdot (1 + 1) + 6 \cdot 9}{9 + (2 + 5 + 2) + \dots + (1 + 1) + 9} = \frac{-17}{100} = -0,17. \quad (1.1)$$

The distance of the centre of mass from the centre of the picture is therefore 0,17 squares.

**2** Marek watches the stream at playback speed  $v_M = 1,75v_S$ , where  $v_S$  is the speed of the streamer's live broadcast. Since the speeds of streamer's and Marek's video do not change, we can use the formula  $v_M = f_M/t_M$ , where  $f_M$  is Marek's current video time and  $t_M$  is the time measured from the moment Marek turned on the video. Similarly, we can write  $v_S = f_S/t_S$ .

Since Marek wants to catch up with the streamer's live broadcast, he has to see exactly what the streamer is broadcasting live. From this we get the condition  $f_S = f_M$ , therefore

$$v_S t = v_M t_M. \quad (2.1)$$

Furthermore, we know that Marek started at time  $t_0 = 30$  min later, that is,  $t = t_0 + t_M$ . By substituting  $t_M$  into the equation 2.1, we get

$$v_S t = v_M (t - t_0). \quad (2.2)$$

Now we substitute  $v_M = 1,75v_S$  and express the time

$$t = \frac{7}{3} t_0 = 70 \text{ min}, \quad (2.3)$$

or, after converting to the hexadecimal system,  $01:10:00$ .

<sup>1</sup>Real speed devils will notice that the topmost and bottommost rows are the same, and will not lose time with them either.

**3** The total altitude difference is the difference between the ascended and descended altitudes. On his way back, the tourist ascends exactly the height he descended before, which is  $1447 - 1302 = 145$  metres.

**4** Apart from a flag, we see three bodies in a thermal contact. They are made from the same sheet metal, so their specific heat capacity  $c$  is the same. And mass? Mass is proportional to their surface area. Using simple geometry we can see that if the flag's mass is  $M$ , the blue part has mass  $\frac{M}{4}$  and each of the two trapezoid parts has mass  $\frac{3M}{8}$ .

From that, we can construct a calorimetric equation in the stable state. The equation states that if all parts of the flag change their temperature to  $T$ , the flag will not receive nor lose any heat:

$$\frac{3M}{8}c(30\text{ °C} - T) + \frac{3M}{8}c(70\text{ °C} - T) + \frac{M}{4}c(90\text{ °C} - T) = 0. \quad (4.1)$$

Now we may simply divide the equation by  $c$  and  $M$  (naturally, this temperature does not depend on neither mass nor material of the flag) and further simplify:

$$\begin{aligned} \frac{3}{8}(30\text{ °C} - T) + \frac{3}{8}(70\text{ °C} - T) + \frac{1}{4}(90\text{ °C} - T) &= 0\text{ °C}, \\ \frac{3}{8} \cdot 30\text{ °C} - \frac{3}{8}T + \frac{3}{8} \cdot 70\text{ °C} - \frac{3}{8}T + \frac{1}{4} \cdot 90\text{ °C} - \frac{1}{4}T &= 0\text{ °C}, \\ T &= 60\text{ °C}. \end{aligned} \quad (4.2)$$

**5** Eight people, each scanning a strip of land 2,5 m wide, can be replaced by a single virtual person scanning a strip of land 20 m wide. The field can be divided into 30 strips, each 10 m wide. We can quickly calculate that it takes  $t = 500$  s for the team to go through two 10 m strips and in the meantime the tractor ploughs five strips. After the time  $t$ , both the team and the tractor are at the other end of the field to their starting point. After each period  $t$  passes, 7 strips have been either searched or ploughed. That means that the team meets the tractor sometime during the 5th period.

In time  $4t$ , they all start from the same side of the field as at the beginning with 28 strips already searched or ploughed. That means that four members of the team would encounter the tractor if they did not stop the search. The team is sad that they have not found anything but at least they can brag that they have searched an area of

$$S_{\text{team}} = 4 \cdot 500\text{ m} \cdot 2 \cdot 10\text{ m} = 40\,000\text{ m}^2. \quad (5.1)$$

Since we are interested in the fraction that could not be searched, we need to express the total area

$$\frac{S_{\text{field}} - S_{\text{team}}}{S_{\text{field}}} = \frac{500\text{ m} \cdot 300\text{ m} - 40\,000\text{ m}^2}{500\text{ m} \cdot 300\text{ m}} = \frac{150\,000 - 40\,000}{150\,000} = \frac{11}{15}. \quad (5.2)$$

**6** Since all our protagonists drive exactly the same way and they abide by the traffic rules, they all start to accelerate exactly at the point where they exit the village. This means that the distances between the cars will not remain the same. However, what does stay the same are the time intervals between their cars passing through any point on the road. This is true, because the movement of each of the cars looks exactly the same except for a slight time delay. We are only interested in the distance between the first and the last car.

Before the cars started to accelerate, their speed had been  $v_0 = 50$  km/h and the distance between the first and the last car had been  $d_0 = 150$  m. This means that the time interval between them was  $t = \frac{d_0}{v_0}$ . After they accelerated to speed  $v_1 = 90$  km/h, this interval remained the same, therefore  $t = \frac{d_1}{v_1}$ . From this we can calculate the new distance between the cars as

$$d_1 = v_1 t = v_1 \frac{d_0}{v_0} = 270 \text{ m.} \quad (6.1)$$

**7** First of all we need to determine which resistance corresponds to which circuit. Two resistors connected in series always have a larger resistance than if connected in parallel. When connected in series, the resistance  $R_S$  is the sum of individual resistances, therefore

$$R_S = R_1 + R_2 = 25 \Omega. \quad (7.1)$$

On the other hand, when connected in parallel, the total resistance  $R_P$  is

$$R_P = \frac{R_1 R_2}{R_1 + R_2} = 4 \Omega. \quad (7.2)$$

Now we have two equations of two unknowns. For example, we can express  $R_1$  as  $R_1 = 25 \Omega - R_2$  and plug it into the second equation. This yields a quadratic equation

$$R_2^2 - 25 \Omega \cdot R_2 + 100 \Omega^2 = 0 \Omega^2. \quad (7.3)$$

Both of its solutions,  $R_2 = 5 \Omega$  and  $R_2 = 20 \Omega$ , are physically meaningful and can be obtained by merely swapping the two resistors. Thus the answer to the question is the pair of values  $5 \Omega$  and  $20 \Omega$ .

**8** Even without the help of the (not very gifted in physics) tourist guide, we know something about the depth of the well – that is

$$h = \frac{1}{2} g t^2 \quad \Rightarrow \quad t = \sqrt{\frac{2h}{g}}. \quad (8.1)$$

The tourist guide tells us the relation with his constant  $k$  (this one, by the way, is not dimensionless)  $h = kt$ . With this information, we can express the constant  $k$  using the relation for the depth of the well as

$$k = \frac{1}{2} g t = \frac{1}{2} g \cdot \sqrt{\frac{2h}{g}} = \sqrt{\frac{hg}{2}}. \quad (8.2)$$

From this we can express  $h$  using only  $k$  and  $g$  as

$$h = \frac{2k^2}{g}. \quad (8.3)$$

**9** At the beginning, there was a mass  $m_{A0}$  of acetone and  $m_{W0}$  of water in the solution. The mass fraction can be obtained as

$$X = \frac{m_{A0}}{m_{A0} + m_{W0}}. \quad (9.1)$$

After Justine returned, there was only  $m_{A1}$  of acetone. One third of the acetone and one tenth of water had evaporated, so

$$m_{A1} = \frac{2}{3}m_{A0} \quad \text{and} \quad m_{W1} = \frac{9}{10}m_{W0}. \quad (9.2)$$

The new mass fraction is

$$\frac{5}{6}X = \frac{m_{A1}}{m_{A1} + m_{W1}}. \quad (9.3)$$

From this equation, we express  $X$  and plug it into the equation 9.1. Then we substitute masses from equation 9.2 to obtain

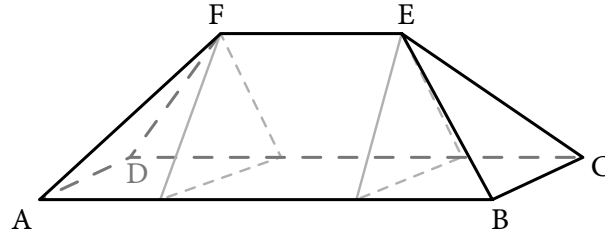
$$\begin{aligned} \frac{m_{A0}}{m_{A0} + m_{W0}} &= \frac{6}{5} \frac{m_{A1}}{m_{A1} + m_{W1}} \\ \frac{\frac{3}{2}m_{A1}}{\frac{3}{2}m_{A1} + \frac{10}{9}m_{W1}} &= \frac{6}{5} \frac{m_{A1}}{m_{A1} + m_{W1}} \\ \frac{3}{2}m_{A1}^2 + \frac{3}{2}m_{A1}m_{W1} &= \frac{6}{5} \frac{3}{2}m_{A1}^2 + \frac{6}{5} \frac{10}{9}m_{A1}m_{W1} \\ \frac{3}{2}m_{A1} + \frac{3}{2}m_{W1} &= \frac{6}{5} \frac{3}{2}m_{A1} + \frac{6}{5} \frac{10}{9}m_{W1} \\ m_{A1} &= \frac{5}{9}m_{W1}. \end{aligned} \quad (9.4)$$

We divided both the equations by  $m_{A1}$ , which is definitely nonzero. Now, let's get back to the equation 9.3, where we can plug  $m_{A1} = \frac{5}{9}m_{W1}$  and see that the result is

$$\frac{\frac{5}{9}m_{W1}}{\frac{5}{9}m_{W1} + m_{W1}} = \frac{5}{6}X \quad \Rightarrow \quad X = \frac{3}{7} \doteq 42,86 \%. \quad (9.5)$$

**10** First we need to realize that the origins of all torques are the plane of a chandelier. While all five candles are on the chandelier, all torques are balanced and the chandelier is in equilibrium. Once a candle falls off, the symmetry of the torques is disturbed and the chandelier will rotate until its centre of mass has minimal altitude. That means that the empty arm will move to the highest possible point, which occurs when the angle between the plane of the chandelier and the horizontal plane is  $90^\circ$ .

**11** If we kept pouring sand in one place, we would create a cone with surface slanted at an angle  $\beta$ . By taking the union of all such cones that can fit onto the parcel we obtain a shape shown in figure 11.1.



11.1. Ábra: Side view

This is a triangular prism truncated on both sides. We can split this shape into three parts – a triangular prism and two half-pyramids that can be merged into a single pyramid with sides slanted at angle  $\beta$ . Let us first calculate its height. Slicing the pyramid with a plane perpendicular to the base creates an isosceles triangle with base  $b$  and two equal angles  $\beta$ . The height of the triangle is then

$$h = \frac{b}{2} \tan \beta. \quad (11.1)$$

Since all faces of the pyramid form an angle  $\beta$  with the horizontal plane, it must be a right pyramid and its base must be a square with side length  $b$ . With this knowledge we can easily deduce that the triangular prism has length  $d = a - b$  and its volume is thus

$$V_h = S_p \cdot d = \frac{hb}{2} d = \frac{b^2}{4} (a - b) \tan \beta. \quad (11.2)$$

The volume of the pyramid can be calculated easily as

$$V_i = \frac{1}{3} S_p h = \frac{b^3}{6} \tan \beta \quad (11.3)$$

and the maximum volume of sand is therefore

$$V = V_h + V_i = \left( \frac{3a - b}{12} \right) b^2 \tan \beta. \quad (11.4)$$

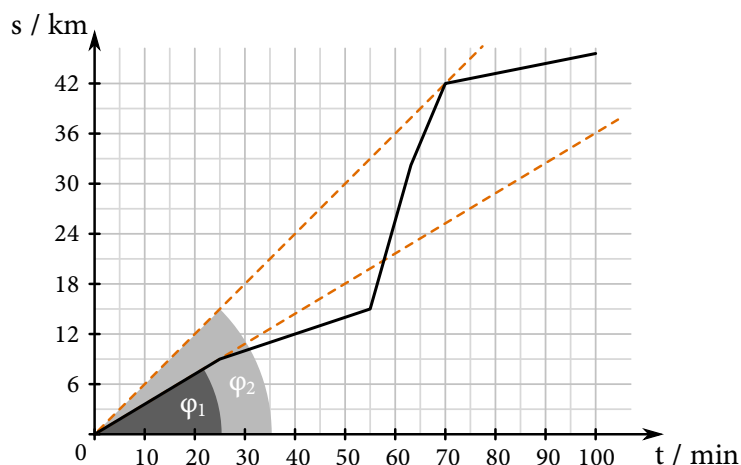
**12** Average speed  $\bar{v}(t)$  at any given time  $t$  is calculated as the distance travelled during the time interval  $t$  divided by the time  $t$ , in our case since  $t = 0$ . Therefore

$$\bar{v}(t) = \frac{s(t)}{t}. \quad (12.1)$$

In other words, we can choose any time  $t$  in the graph and calculate the distance that has been travelled until then. Then we plug these values into the formula for average speed. If we do this for all values of  $t$ , we can simply find the maximum of  $\bar{v}(t)$ . It can, however, be easily seen from the graph in the picture 12.1. Let us modify the formula for average speed so that  $t$  is the independent variable:

$$s(t) = \bar{v}(t)t. \quad (12.2)$$





12.1. Ábra: Angle  $\varphi_1$  is smaller than  $\varphi_2$  which means that  $\bar{v}(t_1) < \bar{v}(t_2)$ .

Now we can see that  $\bar{v}(t)$  is the slope of the line in the graph. Thus, if we want to find the maximum of  $\bar{v}(t)$ , we only need to find the line with the steepest slope, that is, the line for which the angle from the  $x$  axis is the largest. We can see in the graph that this occurs at  $t = 70$  min and it is exactly

$$\bar{v}(t) = \frac{42}{70} \text{ km/min} = 36 \text{ km/h.} \quad (12.3)$$

**13** Since we can distinguish the guests by their names, there are  $5! = 120$  possible permutations of how the tea cups can be assigned. This looks difficult, but we can quickly eliminate a lot of them.

To keep the seesaw balanced, the moments of force from both sides must be equal. Let's denote the saucers  $k_{-2}, k_{-1}, k_0, k_{+1}$  a  $k_{+2}$  and assign the weights 1, 2, 3, 4 and 5 to them. If the distance between two saucers is  $w$  and the mass of the smallest one is  $m$ , in the equation describing moments of force  $w, m$  and  $g$  cancel out and we are left with the equation

$$k_{-2}mg2w + k_{-1}mgw = k_{+1}mgw + k_{+2}mg2w \quad \Big/ \Big/ mgw \quad (13.1)$$

$$2k_{-2} + k_{-1} = k_{+1} + 2k_{+2}.$$

At a distance  $w$  from the centre the masses of cups must have the same parity – if  $k_{-1}$  is odd, then  $k_{+1}$  is also odd and vice versa; otherwise, the moments of force from the left and right side would have different parity. This cannot be balanced out by any combination of cups at  $k_{\pm 2}$ . And the final observation – to every solution we can unambiguously assign its mirror image.

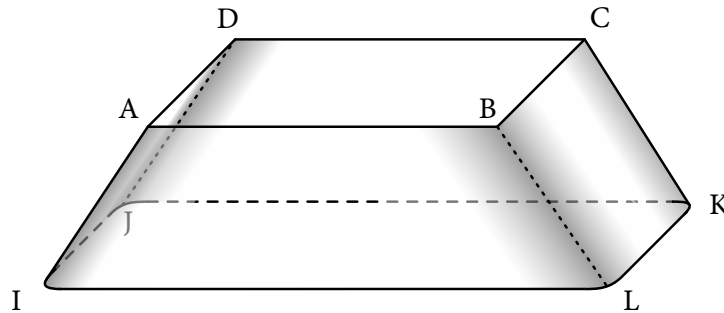
Starting with the heaviest cup, we need to examine three cases:

- If we put it to  $k_{-2}$ , at  $k_{+2}$  we need to place the cup with mass  $4m$  – otherwise, we wouldn't be able to balance the seesaw. From the parity condition, at  $k_{\pm 1}$  we need to place the cups with masses  $m$  and  $3m$ , and the solution is determined as  $5 : 1 : 2 : 3 : 4$  or  $4 : 3 : 2 : 1 : 5$ .
- If we put the heaviest cup to  $k_{-1}$ , from the parity condition we know that  $k_1$  must be occupied by a cup whose mass is an odd multiple of  $m$ . That means we have two options:

- if we put the cup with mass  $m$  there, the difference between  $k_{\pm 2}$  must be  $2m$ , which results in solutions  $2 : 5 : 3 : 1 : 4$  and  $4 : 1 : 3 : 5 : 2$ ;
- if we put the cup with mass  $3m$  there, the difference between  $k_{\pm 2}$  must be  $m$ , which uniquely determines the solutions  $1 : 5 : 4 : 3 : 2$  and  $2 : 3 : 4 : 5 : 1$ .
- Finally, if we place the heaviest cup onto the pivot, the parity condition leaves only two options for us,
  - if positions  $k_{\pm 1}$  are occupied by cups with masses that are even multiplies of  $m$ , we are unable to put  $m$  and  $3m$  cups onto the ends and maintain balance;
  - and in the other case, positions  $k_{\pm 1}$  are occupied by cups with masses that are even multiplies of  $m$ , and again, we quickly find that we cannot balance the seesaw anymore.

We have analyzed all 120 permutations and found that only six of them are valid.

**14** If we dumped all the dirt at one place, we would get a cone with slant angle  $\beta$ . In the image below we see figure we will get by taking the union of all such cones.



14.1. Ábra: Pohľad zhora

Let  $s$  be the distance between the outer edge of our construction and the projection of the monument base. Since the slant angle is  $\beta$  everywhere, this distance will be the same everywhere. We can express it as

$$\cot \beta = \frac{s}{h} \quad \Rightarrow \quad s = h \cot \beta.$$

Now we need to realise that we can divide our figure into several parts. In the corners, we will get four quarters of a cone with radius  $s$  and height  $h$ . Then we have four orthogonal prisms which can be joined into one long prism. Its height is  $2a + 2b$  and its base is a right triangle with sides  $h$  and  $s$ . The last part will be a cuboid with sides  $a \cdot b \cdot h$ . Their volumes are

$$V_{\text{cone}} = \frac{1}{3} \pi s^2 h = \frac{1}{3} \pi h^3 \cot^2 \beta,$$

$$V_{\text{cuboid}} = abh,$$

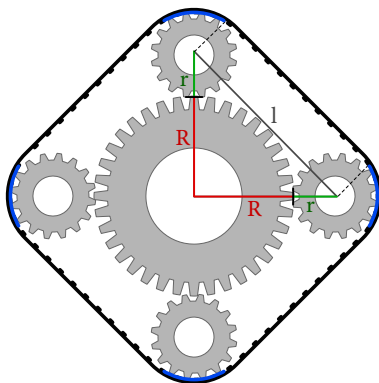
$$V_{\text{prisms}} = S_p(2a + 2b) = \frac{hs}{2}(2a + 2b) = \frac{h^2}{2}(2a + 2b) \cot \beta$$

and the final volume is

$$V = V_{\text{cone}} + V_{\text{cuboid}} + V_{\text{prisms}} = \frac{1}{3} \pi h^3 \cot^2 \beta + abh + h^2(a + b) \cot \beta.$$

After plugging in the values we find out that William will need approximately  $391,27 \text{ m}^3$  of soil.

**15** If there are two cogwheels touching each other, they must have equal tangential speeds, but opposite directions of rotation. This means that all small gears will have equal tangential speeds (same as the tangential velocity of the bigger wheel), which is also the speed with which the belt moves. We can express it as 36 cogs per one rotation of the bigger wheel. Let us denote the width of one cog as  $d$ . Then one rotation of the large cogwheel corresponds to a movement of the belt by  $36d$ . The only thing we need to calculate now is the length of the belt.



15.1. Ábra: Geometry of the belt

In the picture we can see that the belt consists of four arc of  $90^\circ$  which we can put together to make one full circle with circumference  $16d$ . The rest of the belt consists of four line segments of length  $l$ , which is also the distance between the centers of two adjacent small wheels. We can calculate it as the diagonal of a square with side length  $R + r$ , where  $R$  is the radius of the bigger wheel, or  $R = \frac{36d}{2\pi}$ , and  $r$  is the radius of the smaller wheel, or  $r = \frac{16d}{2\pi}$ . Each of the four line segments then has length

$$\sqrt{2}(R + r) = \frac{26\sqrt{2}}{\pi}d. \quad (15.1)$$

The length of the belt is  $(16 + 104\frac{\sqrt{2}}{\pi})d$  and in one revolution of the bigger wheel it moves by  $36d$ . It will therefore take

$$\frac{(16 + 104\frac{\sqrt{2}}{\pi})d}{36d} = \frac{4}{9} + \frac{26\sqrt{2}}{9\pi} \doteq 1,745 \quad (15.2)$$

revolutions of the large cogwheel for the belt to make one revolution.

Let us just add, that if the belt has teeth as well, then its length must be an integer multiple of the width of one cog. Furthermore, if we want all gears to work equally hard, then this integer multiple must be divisible by four. In that case we find that the length of the belt must be  $64d$ , therefore the bigger gear must rotate  $\frac{64d}{36d} \doteq 1,78$ -times.

**16** The atmospheric pressure on Mars is much lower than on the Earth, so the barometer should display a much lower value. However, we must not forget that the barometer measures the relationship between the

atmospheric pressure and the hydrostatic pressure in mercury – which also depends on the magnitude of acceleration due to gravity which is different on Mars.

The gravitational acceleration is given by

$$g_{\sigma} = \frac{GM_{\sigma}}{R_{\sigma}^2}, \quad (16.1)$$

and on small scales the hydrostatic pressure is

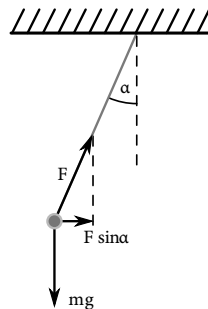
$$p = \rho gh, \quad (16.2)$$

so we can express

$$h = \frac{pR_{\sigma}^2}{\rho GM_{\sigma}}. \quad (16.3)$$

After plugging in the numerical values from the table of constants or other sources we find out that the height of the mercury column is about 12 mm.

**17** To calculate the work we need to know the magnitude of the force  $F$  stretching the string. Let us draw the picture 17.1. Let the string be inclined from the vertical by angle  $\alpha$ . Two forces are now acting on the point mass – force  $\vec{F}$  from the string and the force of gravity  $mg$ . The point is moving with angular velocity  $\omega$  on a circle with radius  $\ell \sin \alpha$ .



17.1. Ábra: Forces acting on the point mass.

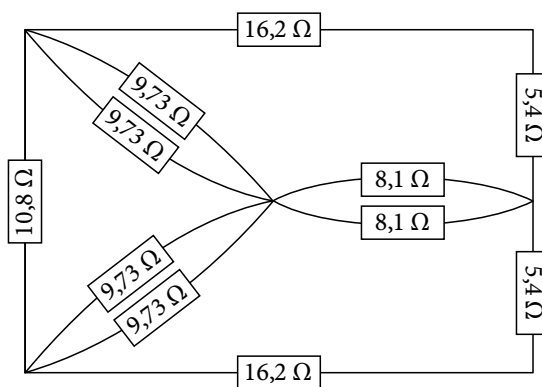
To keep it on a circular trajectory, some centripetal force with magnitude  $m\omega^2 \ell \sin \alpha$  must be act on the point mass. The only force with a horizontal component is the force  $F$  coming from the string, so we may write the equation

$$F \sin \alpha = m\omega^2 \ell \sin \alpha \quad \Rightarrow \quad F = m\omega^2 \ell. \quad (17.1)$$

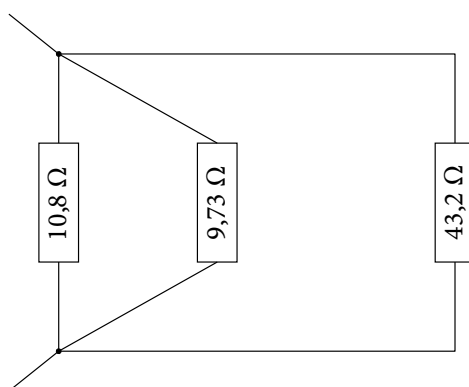
The work Wihlelmina does while pulling the string by a small distance  $\Delta \ell$  is then simply

$$dW = m\omega^2 \ell \Delta \ell. \quad (17.2)$$

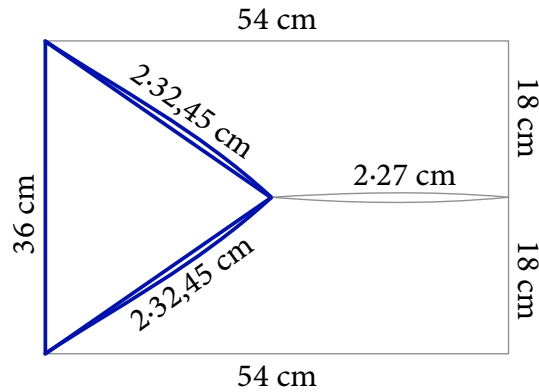
**18** At first, let is calculate the dimensions of the flag and use them to find the resistances of the luminous strips. To find the lengths we need nothing more complicated than the Pythagorean theorem. By multiplying the lengths by resistivity  $30 \Omega/\text{m}$  we obtain their respective resistances, which are displayed in the picture 18.1.


 18.1. Ábra: *Luminous strips as resistors.*

The pair of strips between the white and the red part is interesting – on both of their ends, the potentials are equal. How do we prove this? We could use a simple trick – if we mirror the entire circuit so that the supplying wires are displayed onto each other and the circuit does not change, the points that are displayed onto themselves must have equal potentials. Now, you may ask, is this fact useful for us? Yes, it is! If two points are equal potentials, no current flows between them, even if there is a wire. Hence we may cut the wire and it will not affect the circuit in any way.


 18.2. Ábra: *Wires with current, after their resistance was calculated.*

Now we are left with a combination of resistors connected only in series or in parallel. The voltage drop on every single branch will be equal to the supply voltage, 42 V. We do know their resistances and that means the current in each of the branches can be calculated using Ohm's law,  $I = \frac{U}{R}$ . The branch with resistance of 9,73 Ω is composed from two equal parallel branches; so half of the current will flow through each of them. From left to right, the currents in the branches are 3,89, 2,16, 2,16 és 0,97 A. Currents higher than 2 A – currents blowing a fuse – flow only in the branches displayed as thick lines.



18.3. Ábra: Lengths of luminous strips, thick lines do denote strips with current higher than 2 A.

What is the probability that the technician cuts somewhere in these places? We can calculate that as the ratio of the length of these strips to the sum of all the strips' lengths,

$$P \approx \frac{165,8 \text{ cm}}{363,8 \text{ cm}} \approx 0,456 \doteq 46 \%. \quad (18.1)$$

**19** An emergency brake must slow the train down in the shortest possible distance, and that is determined by the coefficient of friction. The mass of the train remains the same ( $m = 100 \text{ t} + 5 \cdot 20 \text{ t} = 200 \text{ t}$ ), but the braking force can only come from the vehicles whose brakes are already active. It will be equal to  $F = \mu m' g$  where  $m'$  is the total mass of the braking vehicles and  $\mu$  is the coefficient of dynamic friction. The deceleration of the train is then

$$a = \frac{F}{m} = \frac{\mu m' g}{m}. \quad (19.1)$$

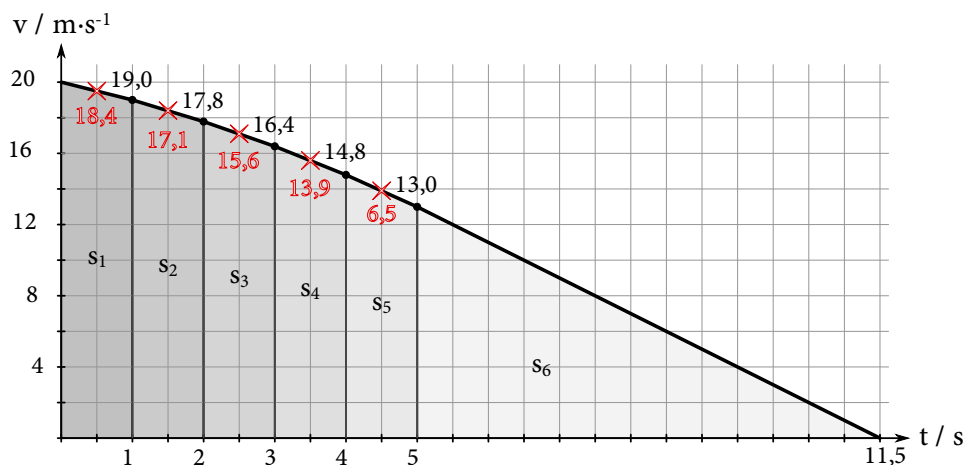
When Lucy activates the brakes, the deceleration will be

$$a_0 = \frac{10^5 \text{ kg}}{2 \cdot 10^5 \text{ kg}} \cdot 0,2 \cdot 10 \text{ m/s}^2 = 1 \text{ m/s}^2. \quad (19.2)$$

After each second, it increases by extra

$$\Delta a = 0,2 \cdot \frac{20 \text{ t}}{200 \text{ t}} \cdot 10 \text{ m/s}^2 = 0,2 \text{ m/s}^2 \quad (19.3)$$

until it reaches its maximal value  $a_{\max} = a_0 + 5 \Delta a = 2 \text{ m/s}^2$  and then train will decelerate at this rate until it comes to a halt. This can be readily expressed in mathematical formula, but during the competition it is probably easier and faster to draw everything (see picture 19.1) and write into the table 19.1.



19.1. Ábra: The speed of the train as a function of time

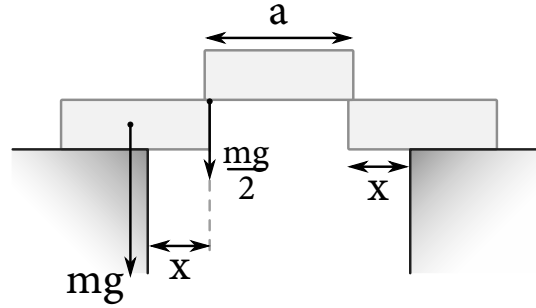
In each interval the train uniformly decelerates from speed  $v_i$  to a lower speed  $w_i = v_i - a_i \Delta t_i$ , which means we can express its average speed in this interval as the mean of the speeds at its beginning and its end. Furthermore,  $v_{i+1} = w_i$ , since its speed has to be continuous. The last interval is special: all vehicles are braking and the train will slow down until it stops. From the speed of  $v_5 = 13$  m/s and at a rate of  $2$  m/s<sup>2</sup> this will take  $6,5$  s and the travelled distance will be  $s_5 = \frac{169}{4}$  m =  $42,25$  m. Finally we express the distance travelled in each interval as the product of its length and the train's average speed in it,  $s_i = \bar{v}_i \cdot \Delta t_i$ .

19.1. Táblázat: Instantaneous decelerations and the corresponding distance.

| number of<br>braking<br>vehicles | $\Delta t_i / \text{s}$ | $F_i / \text{kN}$ | $a_i / \text{m/s}^2$ | $v_i / \text{m/s}$ | $w_i / \text{m/s}$ | $\bar{v}_i / \text{m/s}$ | $s_i / \text{m}$ |
|----------------------------------|-------------------------|-------------------|----------------------|--------------------|--------------------|--------------------------|------------------|
| 0                                | 1                       | 200               | 1,0                  | 20,0               | 19,0               | 19,5                     | 19,50            |
| 1                                | 1                       | 240               | 1,2                  | 19,0               | 17,8               | 18,4                     | 18,40            |
| 2                                | 1                       | 280               | 1,4                  | 17,8               | 16,4               | 17,1                     | 17,10            |
| 3                                | 1                       | 320               | 1,6                  | 16,4               | 14,8               | 15,6                     | 15,60            |
| 4                                | 1                       | 360               | 1,8                  | 14,8               | 13,0               | 13,9                     | 13,90            |
| 5                                | 6,5                     | 400               | 2                    | 13,0               | 0,0                | 6,5                      | 42,25            |

Now we only need to sum everything and we find out that the total braking distance is  $126,75$  m.

**20** First we need to realize, that the blocks must be oriented horizontally, because otherwise if the blocks were tilted at an angle, then due to low friction, the blocks would start slipping and they would fall into the pit. Thus the solution will consist of two blocks placed on the floor, each one protruding into the gap by length  $x$ . The third block will be placed on these two blocks in such a way, that it will only be supported by their edges (see figure 20.1).



20.1. Ábra: *The bridge with maximal span.*

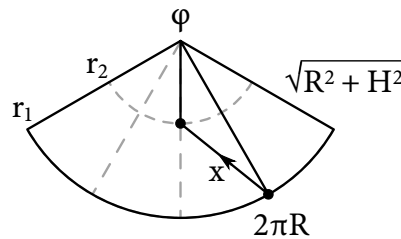
Thus we need to figure out the maximum distance  $x$  by which the blocks can protrude over the gap. For the blocks to not fall into the pit, the total torque produced by forces acting on the lower blocks must be zero. Let us choose one of these lower blocks and for the axis of rotation, we will choose the edge of the gap. The gravitational force acts in the block's center of mass, which is in distance  $a/2 - x$  from the axis, and the upper block acts with force  $mg/2$  in the edge of the block, at distance  $x$  from the axis of rotation. This yields

$$mg\left(\frac{a}{2} - x\right) = \frac{mg}{2}x. \tag{20.1}$$

From this we see that  $x = a/3$  and the total span of the bridge is

$$d = 2x + a = \frac{5}{3}a. \tag{20.2}$$

**21** Findin the minimal distance between two points on a cone may seem difficult, but it is not so, since its surface can be unrolled into a flat plane. The base of the cone is not important in this case and we are left with a disk sector.



21.1. Ábra: *Surface development of the cone.*

As the radius of the base is  $R$ , the curved part of its circumference must be  $2\pi R$  long. Its radius is  $\sqrt{R^2 + H^2}$ , as it is the hypotenuse of a right triangle with sides  $R$  (the radius of the cone's base) and  $H$  (its height). The angle at the apex of the disk sector is  $\varphi = \frac{2\pi R}{\sqrt{R^2 + H^2}}$  and the ant wants to traverse  $90^\circ$  on the cone, which means it will need to travel an angle of  $\varphi/4$  on the developed surface.

On a plane the shortest route is obviously a straight line. If we connect its endpoints with the apex, we obtain a triangle where we can apply the law of cosines. We already know the central angle  $\varphi/4$ , the distance to the beginning is obviously  $r_1 = \sqrt{R^2 + H^2}$  and the distance to the endpoint is  $r_2 = \frac{1}{2}\sqrt{R^2 + H^2}$  since it scales linearly with altitude and the ant wants to get halfway up.



Now let us denote the unknown distance  $x$ . From the cosine law

$$x^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos \frac{\varphi}{4}, \quad (21.1)$$

which, after substituting for  $r_1, r_2$  and  $\varphi$  yields

$$x = \sqrt{R^2 + H^2} \sqrt{\frac{5}{4} - \cos \frac{\pi R}{2\sqrt{R^2 + H^2}}}. \quad (21.2)$$

After plugging in the physical dimensions we see that  $x \doteq 1,56$  m.

**22** Forces applied on a stick are gravitational force pointing downwards and electric forces from other electric charges pointing in different horizontal directions. And there is of course normal force from ceiling, so the whole system is in the state of equilibrium. However, this force is unknown, therefore we need to find other way to solve this problem. The entire system is stationary, that means the sticks aren't rotating and we need to zero the torques.

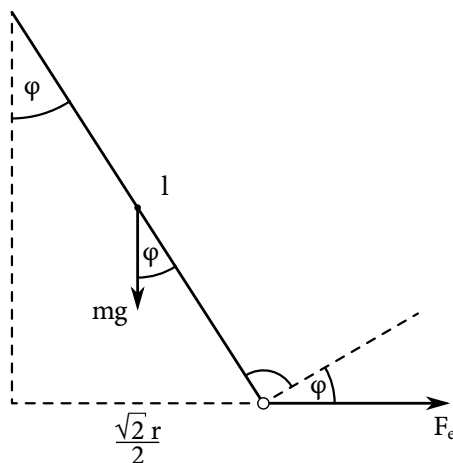
Let us denote electric charge as  $q$ , mass of the stick  $m$  and its length  $d$ . From symmetry of the problem, we can assume that the sticks create a pyramid with a square base with a side length  $r$  and diagonal length  $\sqrt{2}r$ . We can choose any of the four sticks and write equations with forces from nearest charges. Both electric charges have the magnitude

$$F_{e1} = \frac{q^2}{4\pi\epsilon_0 r^2} \quad (22.1)$$

and they are proportional to each other. Therefore, magnitude of the net force is  $\sqrt{2}F_{e1}$  and vector of net force is oriented from the centre of the square to the charge, where the force is applied. Magnitude of the electric force from the farthest charge is

$$F_{e2} = \frac{q^2}{4\pi\epsilon_0 (\sqrt{2}r)^2}. \quad (22.2)$$

So the total net force applied on the charge has magnitude  $F_e = \sqrt{2}F_{e1} + F_{e2}$  and it is oriented from the centre of the square. Gravitational force is oriented downwards and it is applied in the centre of mass of the stick as shown in the figure 22.1.



22.1. Ábra: Sily pôsobiace na jednu paličku.

The bottom segment of the triangle is half of the square diagonal, so its length is  $\frac{\sqrt{2}r}{2}$ . We can also notice from the figure that  $r = \frac{2}{\sqrt{2}}d \sin \varphi$ .

$$\begin{aligned}
 mg \frac{d}{2} \sin \varphi &= \frac{q^2}{4\pi\epsilon_0 r^2} \left( \frac{1}{2} + \sqrt{2} \right) d \cos \varphi \\
 mg \frac{d}{2} \sin \varphi &= \frac{q^2}{4\pi\epsilon_0 \left( \frac{2}{\sqrt{2}} d \sin \varphi \right)^2} \left( \frac{1}{2} + \sqrt{2} \right) d \cos \varphi \\
 \frac{\sin^3 \varphi}{\cos \varphi} &= \frac{q^2}{4\pi\epsilon_0 d^2 mg} \left( \frac{1}{2} + \sqrt{2} \right).
 \end{aligned} \tag{22.3}$$

At this point we see that this problem doesn't have a closed-form solution, so we can't express  $\varphi$  directly. However, we can use calculator and try to guess (wisely) the answer. With a quick binary search we should find out that the answer is  $\varphi \doteq 68^\circ$ .

**23** Let us denote the mass of the lorry  $M$  and the mass of the fly  $m$ . Let the velocity of the lorry be  $v$ , the velocity of the fly  $-v$  and the velocity of both of them after the collision  $u$ . The law of conservation of momentum gives

$$Mv - mv = (M + m)u \quad \Rightarrow \quad u = \frac{M - m}{M + m}v. \tag{23.1}$$

The collision is inelastic so energy is not conserved. The difference between the energy before and after the collision is equal to the heat created by the collision

$$\begin{aligned}
 Q &= \frac{1}{2}M(v^2 - u^2) + \frac{1}{2}m((-v)^2 - u^2) \\
 &= 2 \frac{Mm}{M + m} v^2.
 \end{aligned} \tag{23.2}$$

Mass of the lorry is, obviously, much larger than mass of the fly, therefore we can neglect  $m$  in the denominator. Therefore

$$Q \approx 2mv^2. \tag{23.3}$$

The problem statement says that all heat is spent to increase the former fly, so

$$Q = mc \Delta t. \tag{23.4}$$

After joining the last two equations, we obtain

$$\Delta T \approx \frac{2v^2}{c} \approx 0,13 \text{ }^\circ\text{C}. \tag{23.5}$$

**24** In the beginning, Matthew has zero velocity, therefore zero kinetic energy. However, when he gets to the point with velocity  $v$ , his potential energy decreases by  $mgr$ . Since we don't consider friction and drag

force, his kinetic energy increases by this factor. His velocity is well-known, so we can find the radius from the law of conservation of energy

$$\frac{1}{2}mv^2 = mgr \Rightarrow r = \frac{v^2}{2g}. \quad (24.1)$$

When he gets to the lowest point of the loop his kinetic energy increases again by the  $mgr$ , so  $mv^2 = 2mgr$ . New velocity, we can call it  $w$ , will be  $\sqrt{2}$  times greater. Therefore, we can express magnitude of centripetal force as

$$a_c = \frac{w^2}{r} = \frac{2v^2}{r} = \frac{2v^2 \cdot 2g}{v^2} = 4g. \quad (24.2)$$

Final acceleration is independent of the radius of the loop and his velocity  $v$ . Apart from centripetal force, Matthew also feels reaction from gravitational force with magnitude  $g$ , therefore total  $g$ -force in the lowest point is  $5g$ .

**25** First, let us show what happens if we connect the springs in parallel. The resultant force from the springs is the sum of the forces from the individual springs. At the same time the extension of the individual springs is the same as the extension of the whole system, thus we get

$$\begin{aligned} F &= F_1 + F_2 \\ k \cdot \Delta x &= k_1 \cdot \Delta x + k_2 \cdot \Delta x \\ k &= k_1 + k_2. \end{aligned} \quad (25.1)$$

We can see that if we have two identical springs in parallel, they act like one spring with double the stiffness. If we have  $n$  springs, they will act like a single spring that is  $n$  times stiffer.

We know from the statement that each spring has an unknown rest length, which we will denote as  $s$ . We also know that when Joe hangs on one spring, his distance from the ground is 50 cm – the extension of the spring  $\Delta x_1$  together with the spring's rest length  $s$  is 200 cm. When he hangs on two springs, the extension of the springs  $\Delta x_2$  together with their rest length  $s$  is 110 cm (because it is 140 cm above the ground). And when he hangs on three springs, the extension of the springs is  $\Delta x_3$ . Together with the rest length  $s$  we get the desired distance which we can label as  $\psi$ .

With this notation, we get the equations of force,

$$\begin{aligned} F = mg &= k \Delta x_1 = 2k \Delta x_2 = 3k \Delta x_3, \\ \Delta x_1 &= 2 \Delta x_2 = 3 \Delta x_3, \\ 200 \text{ cm} - s &= 2(110 \text{ cm} - s) = 3(\psi - s). \end{aligned} \quad (25.2)$$

From this we can easily calculate the rest length of the springs

$$200 \text{ cm} - s = 220 \text{ cm} - 2s \quad \Rightarrow \quad s = 20 \text{ cm.} \quad (25.3)$$

Once we know  $s$ , calculating the distance from the ceiling  $\psi$  is trivial

$$3\psi - 3s = 200 \text{ cm} - s \quad \Rightarrow \quad \psi = 80 \text{ cm.} \quad (25.4)$$

To find Joe's distance from the ground, we only need to subtract this from the total height of the room, so the result is  $250 \text{ cm} - \psi = 170 \text{ cm}$ .

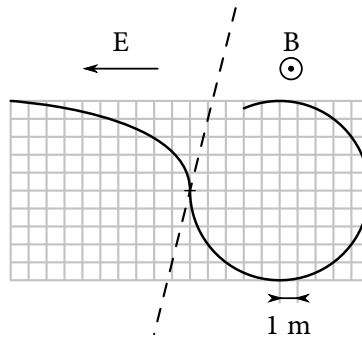
**26** Let's review our knowledge of the forces acting on a charged particle in an electric and in a magnetic field first. In the electric field, the acting force is the electric force

$$\vec{F}_e = Q\vec{E}, \quad (26.1)$$

where  $Q$  is the particle's charge; in a magnetic field, the acting force is the magnetic force

$$\vec{F}_m = Q(\vec{v} \times \vec{B}), \quad (26.2)$$

where  $\vec{v}$  is the particle's velocity. The electric force is either in the direction of the particle or in the opposite direction, depending on the particle charge; the magnetic force is perpendicular to the magnetic field and the particle velocity vector. Concluding from these statements, the electric force can change the particle's speed, while the magnetic force can only alter its direction, i. e. it is a centripetal force.



26.1. Ábra: *The motion of the ball*

Let's analyze the motion of the ball. We will denote the horizontal coordinate as  $x$  and the vertical one as  $y$ . In the figure 26.1, the ball's trajectory is traced, but its direction is not marked. However, a simple algorithm is sufficient for its determination. Let's assume the ball first traverses the magnetic field, then enters the electric field. From the trajectory curvature in the magnetic field it is clear the ball's charge is positive. When a positively charged particle enters the electric field, the electric force and intensity are identically oriented. Therefore the ball is accelerated in the direction to the left, which is in accordance with our observation from 26.1, where the  $x$  coordinate augments together with the  $y$  coordinate. If the ball were to move in the opposite direction, from the traced trajectory in the electric field it would seem the ball is decelerating, so it would be positively charged. However, such an assumption would suggest the curvature should be opposite after the ball enters the magnetic field.

Let the ball's speed be  $v$ . The magnetic field is perpendicular to the trajectory plane, and it will always be perpendicular to the velocity vector, therefore the magnitude of the magnetic force is

$$F_m = QvB. \quad (26.3)$$

The magnetic force is centripetal, therefore

$$QvB = \frac{mv^2}{r} \Rightarrow \frac{Q}{m} = \frac{v}{Br},$$

where  $m$  is the ball's mass. Since  $m$  is constant, the ball's trajectory is a circle. From the 26.1, its radius is  $r = 5$  m. Again, from the direction of the curvature, the charge of the ball is positive.

The ball transits from the magnetic to the electric field. There cannot be any non-uniformity in the change of the velocity vector, as this would mean an infinite force acting in this point. From the figure we see that once the ball enters the electric field, it moves in the direction of the  $y$  coordinate with the same speed it had in the magnetic field,  $v$ . Let's assume the ball stays in the electric field for a time interval  $t$ , during which it traverses a distance of  $\Delta y = 5$  m in direction  $y$ , and a of distance  $\Delta x = 10$  m in the direction  $x$ . There is no acting force in the  $y$  direction, therefore

$$\Delta y = vt;$$

in the  $x$  direction, the acting force is

$$F_e = QE,$$

which describes a uniformly accelerated motion with acceleration

$$a = \frac{QE}{m}$$

and with a zero initial velocity. Therefore

$$\Delta x = \frac{1}{2} \frac{QE}{m} t^2. \quad (26.4)$$

After eliminatng time from these equations we obtain

$$\Delta x = \frac{1}{2} \frac{QE}{m} \left( \frac{\Delta y}{v} \right)^2,$$

in which we recognize a parabola.

Substituting  $\frac{Q}{m}$ , we get

$$\Delta x = \frac{1}{2} \frac{v}{Br} E \left( \frac{\Delta y}{v} \right)^2 \Rightarrow v = \frac{1}{2} \frac{E}{B} \frac{(\Delta y)^2}{r \Delta x}. \quad (26.5)$$

For the numerical values given in the problem statement and seeing the figure 26.1, we can calculate that  $v = 0,5$  m/s.

**27** A full hippo is a solid ball with radius  $r$  and mass  $m$ . The moment of inertia of a ball is

$$I_1 = \frac{2}{5} mr^2. \quad (27.1)$$

What about the hungry hippo? Its shape is a sphere (a full hippo) without a stomach. To find the moment of inertia, we'll have to subtract the moment of inertia of the stomach from the previous result. The stomach is a sphere with mass  $m/8$ , so its radius is  $r/2$ . The moment of inertia of the second hippo is then

$$I_2 = \frac{2}{5}mr^2 - \frac{2}{5} \frac{m}{8} \left(\frac{r}{2}\right)^2 = \frac{31}{80}mr^2. \quad (27.2)$$

Now, let's think of the energy conservation law! Let the hippos lie at an altitude  $h$  over the Nile's water level; that does mean that during their roll the hippos have to descent by  $h$ . The potential energy of the full hippo will decrease by  $mgh$ ; this energy is going to transform into rotational and translational parts of hippo's kinetic energy. Angular speed can be determined as  $\omega = v/r$ , because the hippo is not slipping. For the full hippo we obtain

$$\begin{aligned} mgh &= \frac{1}{2}I_1\left(\frac{v_1}{r}\right)^2 + \frac{1}{2}mv_1^2 \\ &= \frac{1}{5}mv_1^2 + \frac{1}{2}mv_1^2 \\ &= \frac{7}{10}mv_1^2, \end{aligned} \quad (27.3)$$

from which we can express

$$v_1 = \sqrt{\frac{10}{7}gh}. \quad (27.4)$$

Then we do the same for a hungry hippo. We have to remember that this hippo's mass is only  $\frac{7}{8}m$ . Therefore

$$\begin{aligned} \frac{7}{8}mgh &= \frac{1}{2}I_2\left(\frac{v_2}{r}\right)^2 + \frac{1}{2} \frac{7}{8}mv_2^2 \\ &= \frac{31}{160}mv_2^2 + \frac{7}{16}mv_2^2 \\ &= \frac{101}{160}mv_2^2 \end{aligned} \quad (27.5)$$

and from that

$$v_2 = \sqrt{\frac{140}{101}gh}. \quad (27.6)$$

The ratio of these speeds is

$$\frac{v_1}{v_2} = \frac{\sqrt{202}}{14}, \quad (27.7)$$

with the full hippo being faster.

**28** When the ball is being submerged into the lake, the hydrostatic pressure slowly rises and compresses the ball. When the ball shrinks so much that its average density exceeds the density of water, the weight of the ball definitively overtakes the buoyant force and the ball will no longer emerge on its own. Let us denote the mass of the empty ball  $m = 0,1$  kg,  $R_0$  its radius and  $V_0 = \frac{4\pi}{3}R_0^3 \approx 0,0042$  m<sup>3</sup> its volume outside the water.

When Lucy inflates the ball, its total mass increases by the mass of the air to

$$M = m + V_0 \rho_a, \quad (28.1)$$

where  $\rho_a$  is the density of air at standard temperature and pressure. Since Lucy submerges the ball very slowly, the temperature of the gas remains equal to the temperature of water during the entire process. Thus we have an isothermal process, where the equation  $pV = \text{const}$  holds.

The pressure at the water surface is just the atmospheric pressure  $p_0$ . In depth  $h$  under the surface, this pressure increases by hydrostatic pressure  $\rho_w g h$ , where  $\rho_w$  is the density of water. Let us denote the volume of the ball in depth  $h$  as  $V$ . Then the isothermal process yields

$$p_0 V_0 = (p_0 + \rho_w g h) V. \quad (28.2)$$

From this equation we can easily express the volume of the ball in depth  $h$  as

$$V = V_0 \frac{p_0}{p_0 + \rho_w g h}. \quad (28.3)$$

If we want to find the critical depth, we need to set the average density of the ball equal to the density of water, therefore

$$\frac{M}{V} = \rho_w. \quad (28.4)$$

Now we only need to plug the mass  $M$  from equation 28.1 and volume  $V$  of the ball from equation 28.3 into this to get

$$\frac{m + V_0 \rho_a}{V_0 \frac{p_0}{p_0 + \rho_w g h}} = \rho_w \quad (28.5)$$

and express the depth  $h$ . The answer is

$$h = \frac{p_0}{\rho_w g} \left( \frac{4\pi R_0^3 \rho_w}{3m + 4\pi R_0^3 \rho_a} - 1 \right) \approx 392 \text{ m}. \quad (28.6)$$

**29** Since there is no momentum of force acting on the system from the outside of it, the angular momentum is conserved. The energy is not conserved since there is friction and thus some energy is converted into heat. If  $I_1$  is moment of inertia of the disc, the initial angular momentum is  $L = I_1 \omega$ .

After the square with moment of inertia  $I_2$  starts rotating, the total moment of inertia is  $I_1 + I_2$  and the angular frequency drops to  $\omega'$ . The angular momentum is therefore  $L = (I_1 + I_2) \omega'$ . The law of conservation of angular momentum gives

$$\omega' = \frac{I_1}{I_1 + I_2} \omega. \quad (29.1)$$

The remaining part is to plug in the moments of inertia. Moment of inertia of a homogeneous disc  $I_1 = \frac{1}{2} m_1 r^2$  which is  $I_1 = \frac{\pi}{2} \sigma r^4$  when expressed with area density. Moment of inertia of a homogeneous square rotating around its centre can be determined using the parallel axis theorem and the fact that a square comprises of four smaller squares rotating around their corners. The moment of inertia of a square is  $I_2 = K m_2 a^2 = K \sigma a^4$

with an unknown constant  $K$ . It is an additive physical property so it can be expressed as a sum of moments of inertia of four smaller squares with side  $\frac{a}{2}$  and mass  $\frac{m}{4}$  and the rotation axis in the distance  $l = \frac{a}{2\sqrt{2}}$  from their centre of mass. Their constant  $K$  is still the same because they are squares. Using the parallel axis theorem we obtain

$$Km_2a^2 = 4\left(K\frac{m_2}{4}\left(\frac{a}{2}\right)^2 + \frac{m_2}{4}l^2\right), K\sigma a^4 = 4\left(K\sigma\left(\frac{a}{2}\right)^4 + \sigma\left(\frac{a}{2}\right)^2\left(\frac{a}{2\sqrt{2}}\right)^2\right),$$

$$\frac{3}{4}K\sigma a^4 = \frac{1}{8}\sigma a^4, \quad (29.2)$$

$$K = \frac{1}{6}.$$

Moment of inertia of the square is therefore  $I_2 = \frac{1}{6}\sigma a^4$  and the final angular frequency is

$$\omega' = \frac{\frac{\pi}{2}\sigma r^4}{\frac{\pi}{2}\sigma r^4 + \frac{1}{6}\sigma a^4}\omega = \frac{3\pi r^4}{3\pi r^4 + a^4}\omega. \quad (29.3)$$

**30** When electric current flows through a resistor, its temperature increases due to Joule heat. Heat from the resistor is flowing to surrounding air and we know that this heat is proportional to the temperature difference between the resistor and surroundings. The power of the resistor is

$$P_+ = UI = \frac{U^2}{R}. \quad (30.1)$$

and the power of heat flowing away is

$$P_- = kS\Delta T, \quad (30.2)$$

where  $S$  is the area of resistor and  $\Delta T$  is the temperature difference. After a long time, the temperature of the resistor is constant, which means that all heat produced by resistor is flowing to the surroundings, so we can write  $P_+ = P_-$

$$\frac{U^2}{R} = kS\Delta T \Rightarrow \Delta T = \frac{U^2}{RkS}. \quad (30.3)$$

Moreover, we can express the resistance of a conductor with length  $\ell$  and cross-sectional area  $S$  as

$$R = \rho\frac{\ell}{S}, \quad (30.4)$$

where  $\rho$  is the resistivity of material. If we increase the proportions of the conductor by factor  $\alpha$ ,  $\ell$  also increases by factor  $\alpha$  and  $S$  increases by  $\alpha^2$ . This scaling is independent of the shape of the conductor, that means if we had a conductor with different cross-section areas, every cross-section area would still scale by factor  $\alpha$ . Therefore, we can think of the resistor as of an imperfect conductor. If we scale its size by factor  $\alpha$ , its resistivity always decreases by  $\alpha$ .

After a long time, the scaled resistor reaches its equilibrium temperature  $T_2$  and the temperature difference with respect to the surrounding medium is  $\Delta T_2 = T_2 - T_0$ . With this knowledge we can simplify equation



## 30.3

$$\Delta T_2 = \frac{U^2}{\frac{R}{\alpha} k \alpha^2 S} = \frac{1}{\alpha} \frac{U^2}{RkS} = \frac{1}{\alpha} \Delta T. \quad (30.5)$$

Moreover, we know that  $\alpha = 2$ , so  $\Delta T_2 = \frac{1}{2} \Delta T$ . The temperature difference for the small resistor is  $\Delta T = T_1 - T_0$ , and for the big resistor it is  $\Delta T = T_2 - T_0$ , from where we can express

$$T_2 = \frac{T_0 + T_1}{2}.$$

**31** The moment of inertia of the star can be calculated as the sum of all the moments of inertia from the individual rods about the axis passing through the centre of the star. The moment of inertia of a single rod with mass  $m$  and length  $d$  about its center of mass is  $I_0 = \frac{1}{12} md^2$ . If we want to calculate the moment of inertia about a different axis, we can use the parallel axis theorem, which states that if the new axis is parallel to the original axis, the new moment of inertia can be calculated as

$$I = I_0 + mx^2, \quad (31.1)$$

where  $x$  is the distance between the two axes.

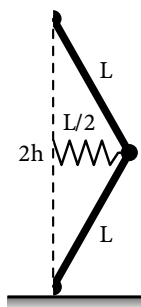
In case of the star, the distance between the axis passing through the center of the rod and the axis passing through the center of the star is one third of the length of the median of the equilateral „triangles“ that the star is made out of. The length of the median of such a triangle can be determined using the Pythagorean theorem as

$$a = \sqrt{d^2 - \left(\frac{d}{2}\right)^2} = \frac{\sqrt{3}}{2} d. \quad (31.2)$$

The moment of inertia of the star about its center is then

$$I = 6 \left( I_0 + m \left( \frac{a}{3} \right)^2 \right) = 6 \left( \frac{1}{12} md^2 + \frac{1}{12} md^2 \right) = md^2. \quad (31.3)$$

**32** The stiffness of the spring in Dave's work of art can be calculated using the method of virtual work. It states that if the system is displaced from its equilibrium position by an infinitesimal amount, its energy does not change. The work of art is symmetric around the vertical axis, therefore it is enough if we look on one of its halves, the other must remain symmetric.



32.1. Ábra: Proportions of the „artwork“ in equilibrium position

Dave's system is in its equilibrium when the spring is stretched to length  $L$ . Then we can calculate the height of the system (let us denote it as  $2h$ ) using the Pythagorean theorem

$$L^2 = \left(\frac{L}{2}\right)^2 + h^2, \quad \Rightarrow \quad h = \frac{\sqrt{3}}{2}L. \quad (32.1)$$

The total potential energy of the system can be calculated as the sum of gravitational potential energies of the four sticks and the potential energy of the spring with stiffness  $k$ . In equilibrium, the center of mass of the two sticks is at height  $\frac{h}{2}$  and for the other two sticks at height  $\frac{3h}{2}$ . Therefore the energy is

$$E = 2\left(\frac{1}{2}mgh + \frac{3}{2}mgh\right) + \frac{1}{2}kL^2 = 4mgh + \frac{1}{2}kL^2. \quad (32.2)$$

Now suppose that the spring is stretched by an infinitesimal distance  $dx$ . Since the length of the sticks cannot change, the height  $h$  will change to  $h + \Delta h$ , while the Pythagoras theorem must still hold:

$$L^2 = \left(\frac{L + dx}{2}\right)^2 + (h + dh)^2. \quad (32.3)$$

Neglecting the infinitesimal changes of second order, i. e.  $(dx)^2$  and  $(dh)^2$ , yields a relation between the height and the length of the spring

$$dh = -\frac{L}{4h} dx. \quad (32.4)$$

Now let us look on the energy, where we are again neglecting the infinitesimal changes in second order

$$E = 4mg(h + dh) + \frac{1}{2}k(L + dx)^2 \approx 4mg(h + dh) + \frac{1}{2}k(L^2 + 2L dx). \quad (32.5)$$

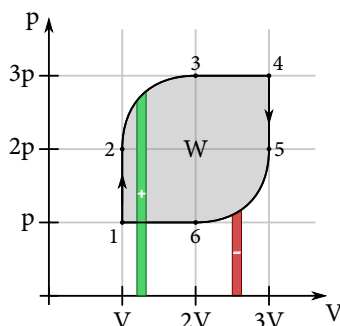
Since the energy cannot change, we get

$$0 = 4mg dh + kL dx = \left(-mg\frac{L}{h} + kL\right) dx. \quad (32.6)$$

Rearranging and plugging the height  $h$  into the equation gives us the stiffness of the spring,

$$k = \frac{mg}{h} = \frac{2}{\sqrt{3}} \frac{mg}{L}. \quad (32.7)$$

**33** Let us denote the significant states of the gas by numbers from 1 to 6, as shown in the figure 33.1. The lowest temperature of the gas is in point 1 and the highest is in state 4. Let the temperature in state 1 be  $T_1 = T$ . Then we can calculate the temperature in any other point using the ideal gas law, hence for any of these significant states we know the volume, pressure, and temperature.


 33.1. Ábra:  $pV$  diagram so zakreslenými význačnými stavmi.

We are interested in the efficiency of the engine. That is defined as the ratio of the work gained from the cycle to the total heat supplied into the gas

$$\eta = \frac{W}{Q_{\text{in}}}. \quad (33.1)$$

Let us begin with the work. With a small change in volume  $\Delta V$  the work done by gas equals  $\Delta W = p \Delta V$ . Geometrically, on the  $pV$ -diagram, it corresponds to the area of a small rectangle with width  $\Delta V$  stretching from the curve to the  $V$ -axis. The total work done by the gas between two states is therefore equal to the area under the curve between these two states. The work between states 1 and 4 is positive, i. e. work is done by the gas, and the work between the states 4 and 1 is negative, which means that the work is done on the gas. The total work done by the gas is then equal to the area enclosed by the curve<sup>2</sup>

$$W = \left(2 + \frac{\pi}{2}\right)pV. \quad (33.2)$$

Now let's look on the supplied heat. The first law of thermodynamics tells us that the heat supplied to the system will be spent on work and change in internal energy

$$Q = W + \Delta U. \quad (33.3)$$

However, the internal energy is directly proportional to temperature. Each molecular degree of freedom contributes to the internal energy by amount  $\frac{1}{2}kT$ , and since all of these molecules have three degrees of freedom<sup>3</sup> and there are  $N$  molecules, so

$$\Delta U = \frac{3}{2}Nk \Delta T. \quad (33.4)$$

We only supply heat to the cycle between states 1 and 4. The temperature in state 4 is by the ideal gas law equal to

$$T_4 = \frac{p_4 V_4}{p_1 V_1} T_1 = \frac{3p}{p} \frac{3V}{V} T = 9T. \quad (33.5)$$

<sup>2</sup>Two rectangles  $p \times V$  and two quarterellipses with semi axes  $p$  a  $V$ .

<sup>3</sup>The gas is monoatomic, therefore has three translational degrees and zero rotational.

The change in internal energy is then

$$\Delta U_{14} = \frac{3}{2}Nk(T_4 - T_1) = 12NkT \quad (33.6)$$

and the work done by the gas between these two states is

$$W_{14} = \left(5 + \frac{\pi}{4}\right)pV. \quad (33.7)$$

The total supplied heat is then

$$Q_{\text{in}} = W_{14} + \Delta U_{14} = \left(5 + \frac{\pi}{4}\right)pV + 12NkT. \quad (33.8)$$

Using the ideal gas law  $pV = NkT$  we get

$$Q_{\text{in}} = \left(17 + \frac{\pi}{4}\right)pV. \quad (33.9)$$

If we plug this into the definition of efficiency, we get

$$\eta = \frac{\left(2 + \frac{\pi}{2}\right)}{\left(17 + \frac{\pi}{4}\right)} \doteq 0,2 = 20 \%. \quad (33.10)$$

**34** Geometrically, the problem is not that complicated, but the calculation is going to be a bit harder. From the picture, we see that for a real distance of two parsecs

$$2 \text{ pc} = \frac{2 \text{ au}}{\tan 1''} \quad (34.1)$$

and for Martin's unit  $\mathcal{P}$

$$\mathcal{P} = \frac{1 \text{ au}}{\tan 0,5''}. \quad (34.2)$$

We are interested in the difference

$$\mathcal{P} - 2 \text{ pc} = \frac{1 \text{ au}}{\tan 0,5''} - \frac{2 \text{ au}}{\tan 1''}. \quad (34.3)$$

And here is the problem.  $1''$  is a really small angle and the number of metres in one parsec is really huge. When subtracting two numbers that differ somewhere around the twelfth digit, the calculator rounds them in its memory and provides only a result which is really inaccurate, or it may even tell us it is zero.<sup>4</sup>

Let us use the Taylor expansion of the tangent function:

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots \quad (34.4)$$

<sup>4</sup>A computer can do it. But you are not supposed to have one during Náboj. A good calculator may be able to calculate it too – if you have one, you're done here.

Considering the required accuracy of 1 km, a rough estimate tells us that using the first different term of expansion should be sufficient. Using this third-order term, we get

$$2 \text{ pc} \approx \frac{1}{x + \frac{x^3}{3}} \cdot 2 \text{ au} \quad \text{and} \quad \mathcal{P} \approx \frac{1}{\frac{x}{2} + \frac{(\frac{x}{2})^3}{3}} \cdot 1 \text{ au}, \quad (34.5)$$

where  $x$  is  $1''$ .

After some simplification, we obtain

$$\mathcal{P} - 2 \text{ pc} \approx \frac{2}{x} \left( \frac{1}{1 + \frac{x^2}{12}} - \frac{1}{1 + \frac{x^2}{3}} \right) \cdot 1 \text{ au}. \quad (34.6)$$

After these Taylor atrocities we will lose nothing if we also linearize

$$\frac{1}{1+x} \approx 1-x, \quad (34.7)$$

which lets us simplify the equation 34.3 to

$$\mathcal{P} - 2 \text{ pc} \approx 1 \text{ au} \cdot \frac{2}{x} \left( 1 - \frac{x^2}{12} - 1 + \frac{x^2}{3} \right) = 1 \text{ au} \cdot \frac{2}{x} \frac{x^2}{4} = 1 \text{ au} \cdot \frac{x}{2}. \quad (34.8)$$

Now we plug  $x = 1'' = \frac{\pi}{180 \cdot 60 \cdot 60} = \frac{\pi}{648000}$  back into the equation and obtain the result

$$\mathcal{P} - 2 \text{ pc} \approx \frac{1''}{2} \cdot 1 \text{ au} = \frac{\pi}{1296000} \cdot 1,5 \cdot 10^8 \text{ km} \doteq 364 \text{ km}. \quad (34.9)$$

If we include the next term ( $\propto x^5$ ) we would get a more precise result, but only by about  $2 \mu\text{m}$ . Finally let us remark that the modern definition of parsec does not use a tangent of an angle anymore. but defines parsec explicitly as  $1 \text{ au} \cdot \frac{648000}{\pi}$ .

**35** The resulting temperature of the reflective sphere in the equilibrium state does not depend at all on what the black body inside looks like, at what temperature it would be when not covered, or how much of the radiation the outer sphere reflects back in: if the energy source supplies the power  $P$ , this energy must simply also be emitted again. The foil has two surfaces – inner and outer – and their total area is  $2S$ . Only half of this area radiates out, so

$$P = \frac{1}{2} \sigma 2S \tilde{T}^4,$$

where  $\tilde{T}$  is its temperature. Before the foil was installed, in equilibrium state we had  $P = \sigma S T^4$ . Therefore  $\tilde{T} = T$ .

**36** First of all, the mirror is radially symmetric so we can solve this problem only in two dimensions with two coordinates, namely  $z$  (height) and  $r$  (distance from the rotation axis). The coordinate system will be rotating with the mirror. Let us look at the forces acting on a small element of mercury. Reduced to unit mass, they are

- gravitational acceleration  $\vec{a}_g$  pointing down with magnitude  $g$ ;

- centrifugal acceleration  $\vec{a}_n$  pointing away from the rotation axis, whose magnitude increases linearly with  $r$  as  $\omega^2 r$ .

Their corresponding potentials are  $gz$  and  $-\frac{1}{2}\omega^2 r^2$ . The total potential is therefore

$$U(r, z) = gz - \frac{1}{2}\omega^2 r^2. \quad (36.1)$$

Mercury will form a shape which has constant potential on the whole surface (if the potential wasn't constant, mercury would want to reshape); and since potential is defined up to an additive constant, we can choose it so that the potential is zero on the surface of the mercury. The shape of the surface is thus given by a line whose explicit equation we can obtain by

$$gz - \frac{1}{2}\omega^2 r^2 = 0 \quad \Rightarrow \quad z = \frac{\omega^2}{2g} r^2. \quad (36.2)$$

Here we recognise a parabola – now we only need to determine its parameters. All light rays parallel to the axis of the mirror (vertical rays in this problem) will concentrate in one point (which is a pleasant finding given that we are building a telescope). The simplest way to find the focus is to find a ray that will be horizontal after reflecting from the parabola. In the point where this particular ray hits the mirror, the mercury makes a  $45^\circ$  angle with the horizontal line and the  $z$  coordinate of the point is the same as that of the focus point. Or we can recall that a parabola is a set of points which are at the same distance from the focus and the directrix and that in the point where the particular ray hits the mirror we know that  $z = f$  and  $r = 2f$ . Then

$$f = \frac{\omega^2}{2g} 4f^2 \quad \Rightarrow \quad f = \frac{g}{2\omega^2}. \quad (36.3)$$

**37** At the moment when the cylinder is closed, the gas pressure is  $p_{\text{atm}}$  and the volume of the gas inside is  $SH$ , where  $S$  is the area of the cylinder's base and  $H$  is its height. After reaching equilibrium, the piston is at height  $h$ . That means that the pressure inside is increased by  $\frac{kh}{S}$  by the spring acting on the piston. Since we are interested in the equilibrium height  $h$  after a long time, temperature of the gas will be the same as at the beginning so we consider this as isothermic compression which means that

$$p_{\text{atm}} SH = \left( p_{\text{atm}} + \frac{kh}{S} \right) Sh. \quad (37.1)$$

The equilibrium height  $h$  is therefore<sup>5</sup>

$$h = -\frac{Sp_{\text{atm}}}{2k} + \sqrt{\left( \frac{Sp_{\text{atm}}}{2k} \right)^2 + \frac{SHp_{\text{atm}}}{k}}. \quad (37.2)$$

The equilibrium pressure inside is

$$p = p_{\text{atm}} + \frac{kh}{S} = \frac{p_{\text{atm}}}{2} + \sqrt{\left( \frac{p_{\text{atm}}}{2} \right)^2 + \frac{kHp_{\text{atm}}}{S}}. \quad (37.3)$$

<sup>5</sup>After solving the quadratic equation we utilise the positive solution.

Now let's displace the piston by  $x$ . This displacement results in decrease in pressure by  $\Delta p$ . We are interested in the apparent stiffness in the very first moment when the gas and the surroundings have not yet exchanged any heat. Thus we consider this to be an adiabatic process which is described by

$$p(Sh)^\kappa = (p - \Delta p)[S(h + x)]^\kappa. \quad (37.4)$$

Hence

$$\Delta p = p - \frac{ph^\kappa}{(h + x)^\kappa}. \quad (37.5)$$

The force acting on a piston is

$$F = -\Delta pS - kx = -pS\left[1 - \left(1 + \frac{x}{h}\right)^{-\kappa}\right] - kx \quad (37.6)$$

and for small displacements

$$F = -\left(\frac{\kappa pS}{h} + k\right)x, \quad (37.7)$$

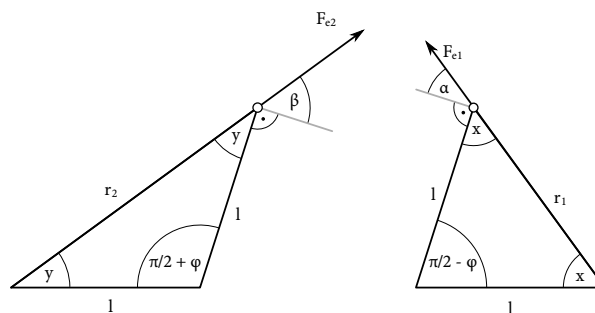
whence we immediately see that the apparent stiffness is

$$K = -\frac{F}{x} = \frac{\kappa pS}{h} + k = \left(\frac{2\kappa}{\sqrt{1 + \frac{4kH}{Sp_{\text{atm}}}} - 1} + \kappa + 1\right)k. \quad (37.8)$$

For the numerical values from the problem statement the result is  $K \approx 1,8 \text{ kN/m}$ .

**38** Let's denote the charge on Paula's sticks  $q$ , their length  $d$  and mass  $m$ . We are interested in small oscillations, i. e. small angular displacement  $\varphi$  from the equilibrium (which is obviously the position when the stick is standing upwards) so if we encounter any formula containing  $\varphi$ , we can make use of the Taylor series and then neglect all terms with  $\varphi^2$  and higher orders.

Three forces act upon the middle stick – gravitational and two electric – and since the stick can rotate around its bottom end, we want to know the corresponding torques. The gravitational force of magnitude  $mg$  acts in the middle of the stick so the corresponding torque at angular displacement  $\varphi$  is  $\frac{d}{2}mg \sin \varphi \approx \frac{d}{2}mg\varphi$ , where displacement  $\varphi$  in clockwise direction is considered positive and torque is positive if it is trying to turn the stick in clockwise direction.



38.1. Ábra: A displaced stick and distances and angles in the triangles.

It gets a bit complicated with moments of electrical forces. Let the distance of the upper point charge from the one on the table on the right be  $r_1$ . Applying the law of cosines to the triangle in the right half of the picture 38.1 yields

$$r_1^2 = d^2 + d^2 - 2dd \cos\left(\frac{\pi}{2} - \varphi\right) = 2d^2(1 - \sin \varphi) \approx 2d^2(1 - \varphi). \quad (38.1)$$

The square of the distance to the left charge is analogically  $r_2^2 = 2d^2(1 + \varphi)$ . If we know the distances, we know the magnitudes of the electrical forces, but we also want to know their directions. Let's look at the picture 38.1 once again.

We have two isosceles triangles which we can use to calculate the angles

$$x = \frac{\pi}{4} + \frac{\varphi}{2} \quad \text{a} \quad y = \frac{\pi}{4} - \frac{\varphi}{2}, \quad (38.2)$$

because the sum of interior angles in a triangle is  $\pi$ . Furthermore, we also see that

$$\alpha + \frac{\pi}{2} + x = \pi \quad \text{a} \quad \beta + \frac{\pi}{2} + y = \pi, \quad (38.3)$$

so

$$\alpha = \frac{\pi}{4} - \frac{\varphi}{2} \quad \text{a} \quad \beta = \frac{\pi}{4} + \frac{\varphi}{2}. \quad (38.4)$$

To calculate the torque, we need to know cosines of these angles, so

$$\cos \alpha = \cos \frac{\pi}{4} \cos \frac{\varphi}{2} + \sin \frac{\pi}{4} \sin \frac{\varphi}{2} \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \frac{\varphi}{2} \quad (38.5)$$

and analogically

$$\cos \beta = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \frac{\varphi}{2}. \quad (38.6)$$

Finally we are getting to the torques where we use Taylor series  $\frac{1}{1-\varphi} \approx 1+\varphi$  and  $\frac{1}{1+\varphi} \approx 1-\varphi$ . Angular acceleration  $\varepsilon$  is then calculated as

$$\begin{aligned} J\varepsilon &= -d \frac{q^2}{4\pi\varepsilon_0 r_1^2} \cos \alpha + d \frac{q^2}{4\pi\varepsilon_0 r_2^2} \cos \beta + \frac{d}{2} mg \sin \varphi \\ &\approx -d \frac{q^2}{4\pi\varepsilon_0 2d^2} (1+\varphi) \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \frac{\varphi}{2} \right) + d \frac{q^2}{4\pi\varepsilon_0 2d^2} (1-\varphi) \left( \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \frac{\varphi}{2} \right) + \frac{d}{2} mg \varphi \\ &\approx - \left( \frac{q^2}{4\pi\varepsilon_0} \frac{3\sqrt{2}}{4d} - \frac{d}{2} mg \right) \varphi. \end{aligned} \quad (38.7)$$

The moment of inertia of a stick rotating around its end is  $J = \frac{1}{3} md^2$ , which, after plugging in into previous equation, yields

$$\varepsilon = - \frac{3}{md^2} \left( \frac{q^2}{4\pi\varepsilon_0} \frac{3\sqrt{2}}{4d} - \frac{d}{2} mg \right) \varphi, \quad (38.8)$$



which is a simple harmonic oscillator with angular frequency

$$\omega^2 = \frac{3}{md^2} \left( \frac{q^2}{4\pi\epsilon_0} \frac{3\sqrt{2}}{4d} - \frac{d}{2} mg \right). \quad (38.9)$$

Thus, the period of small oscillations with the numerical values from the problem statement is

$$T = \frac{2\pi}{\omega} \doteq 1,95 \text{ s}. \quad (38.10)$$

**39** The light bulb shines equally in all directions, so we are able to express its luminous intensity  $I$  as a fraction of its total flux  $\Phi$ ,

$$I = \frac{\Phi}{4\pi}. \quad (39.1)$$

Apart from this we will also need to know its *luminous emittance*, or the luminous flux per unit surface area when looking along any ray ending on its surface. If we denote its radius  $r$ , the area of its cross-section is  $\pi r^2$  and emittance

$$L = \frac{I}{\pi r^2} = \frac{\Phi}{4\pi^2 r^2}. \quad (39.2)$$

How does it look under the lamp? The reflector redirects a fraction of the light onto the pavement, increasing the illumination in comparison to a naked bulb. Since the bulb is exactly in the focus of the paraboloid, the intensity will increase markedly along its axis. The bulb is not a point source, so these rays will not be perfectly parallel – however if we are looking from a place sufficiently far below the lamp, where the angles in question are already very small, we can look in any direction and see

- either the sky or the lamppost (which are of no interest to us since they do not emit any light);
- or the bulb (and therefore emittance  $L$ );
- or its reflection from the polished surface of the reflector (and again emittance  $L$ ).

For this ratio of sizes of the bulb and the mirror and the height of the lamp this condition certainly holds. On the whole, when looking from below the lamp behaves as if it had a larger bulb with the same diameter as the reflector, but with same emittance as the real bulb<sup>6</sup> Now we only need to find the illuminance, which can be done by finding the solid angle which the reflector subtends when looking from the surface of the road, and multiply it by luminance. Since all angles are small, we can approximate the solid angle as

$$\Omega \approx \pi \left( \frac{R}{H} \right)^2, \quad (39.3)$$

where  $R$  is the radius of the reflector and  $H \gg R$  the height of the lamppost. The illuminance under the lamp will then be

$$E = L_0 \Omega \approx \frac{\Phi}{4\pi^2 r^2} \pi \left( \frac{R}{H} \right)^2 \approx 354 \text{ lx}. \quad (39.4)$$

<sup>6</sup>The total luminous flux is of course conserved: only light that would have been emitted upwards or to the sides, is now reflected down. Only light that reflects back onto the bulb cannot exit the reflector.

**40** Kepler's laws tell us that if we throw an object in radial gravitational field of the Moon, it will follow an elliptical trajectory with one of its foci  $F_1$  in the center of the Moon. We only need to find the position of the second focus point so that the initial speed will be minimal. Let us denote the angular distance between Matthew and Jacob as  $\varphi$  (in our case,  $\varphi = 90^\circ$ , because we are throwing the rock from the North Pole to the equator). It is clear that the other focus point will lie on the bisector of this angle (see figure 40.1). To find this point we can use the *vis-viva equation*.

It relates the total energy of an object orbiting a planet with the semi-major axis of the ellipse. We will use it in form

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a} \quad (40.1)$$

where the expression on the left-hand side is the total energy of the orbiting body (kinetic and potential) and  $a$  is the semi-major axis of its orbit.

The derivation of this equation relies on using the conservation laws for energy and angular momentum. Consider a point mass orbiting a planet of mass  $M$  and let us denote its speed at the pericentre as  $v_p$  and its distance from the planet as  $r_p$ ; and at the apocentre  $v_a$  and the distance  $r_a$  respectively. The energy conservation law tells us that

$$\frac{1}{2}mv_p^2 - G\frac{Mm}{r_p} = \frac{1}{2}mv_a^2 - G\frac{Mm}{r_a} \quad (40.2)$$

and the angular momentum conservation law yields

$$mv_p r_p = mv_a r_a \quad \Rightarrow \quad v_a = v_p \frac{r_p}{r_a}. \quad (40.3)$$

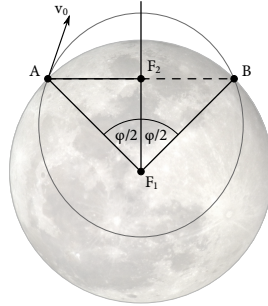
If we plug this into the equation 40.2, we get

$$\frac{1}{2}mv_p^2 - G\frac{Mm}{r_p} = \frac{1}{2}mv_p^2 \frac{r_p^2}{r_a^2} - G\frac{Mm}{r_a} \quad \Rightarrow \quad \frac{1}{2}mv_p^2 \left(1 - \frac{r_p^2}{r_a^2}\right) = GMm \left(\frac{1}{r_p} - \frac{1}{r_a}\right). \quad (40.4)$$

Finally, dividing by the expression  $\left(1 - \left(\frac{r_p}{r_a}\right)^2\right)$  and using  $2a = r_a + r_p$  gives us the result

$$\begin{aligned} \frac{1}{2}mv_p^2 &= \frac{GMm}{2a} \frac{r_a}{r_p} \\ \frac{1}{2}mv_p^2 - G\frac{Mm}{r_p} &= -\frac{GMm}{2a}. \end{aligned} \quad (40.5)$$

The law of conservation of energy says that the expression  $mv^2/2 - GMm/r$  is constant along the object's trajectory. We have managed to express it at the point of its closest approach, therefore it must be the same for any other point on the trajectory. Thus the vis-viva equation (40.1) is proven.


 40.1. Ábra: *The trajectory of the rock.*

Let us denote the unknown initial speed of the rock as  $v_0$ . Then the equation 40.1 becomes

$$\frac{1}{2}mv_0^2 - G\frac{Mm}{R} = -G\frac{Mm}{2a} \Rightarrow v_0^2 = GM\left(\frac{2}{R} - \frac{1}{a}\right). \quad (40.6)$$

Notice that the initial speed  $v_0$  decreases with decreasing semi-major axis. Therefore we only need to find the position  $F_2$  of the second focus so that the length of the semi-major axis is minimal. Next we notice that if we take any point on the ellipse, e. g. point A, and we calculate the sum of the distances from this point to the foci of the ellipse ( $|AF_1| + |AF_2|$  in the figure), we always get  $2a$ . However, the distance  $|AF_1|$  is always equal to the radius of the Moon  $R_\oplus$ , so we can only minimize  $|AF_2|$ .

The shortest possible distance corresponds to the situation when point  $F_2$  lies on the line connecting A and B. Then

$$2a = |AF_1| + |AF_2| = R_\oplus \left(1 + \sin \frac{\varphi}{2}\right) \quad (40.7)$$

and if we plug this into equation 40.6 and express the speed  $v_0$ , we get

$$v_0 = \sqrt{\frac{2GM_\oplus}{R_\oplus} \left(1 - \frac{1}{1 + \sin \frac{\varphi}{2}}\right)}. \quad (40.8)$$

Setting  $\varphi = 90^\circ$  yields

$$v_0 = \sqrt{\frac{GM_\oplus}{R_\oplus} \frac{2\sqrt{2}}{2 + \sqrt{2}}}. \quad (40.9)$$

# Válaszok

1 Pontosan 0,17.

2 01:10:00

3 145 m

4 60 °C

5  $\frac{11}{15}$

6 270 m

7 5 Ω és 20 Ω, bármilyen sorrendben.

8  $h = \frac{2k^2}{g}$

9  $\frac{3}{7}$

10 90°

11  $\left(\frac{3a-b}{12}\right)b^2 \tan \beta$

12 36 km/h

13 6

14 391 m<sup>3</sup>

15  $\frac{4}{9} + \frac{26\sqrt{2}}{9\pi} = \frac{4\pi + 26\sqrt{2}}{9\pi}$

16 12 mm

17  $dW = m\omega^2 \ell d\ell$

$$\boxed{18} \quad 2 \frac{1 + \sqrt{13}}{13 + 2\sqrt{13}} = 2 \frac{11\sqrt{13} - 13}{117} \doteq 46 \%$$

$$\boxed{19} \quad 126,75 \text{ m}$$

$$\boxed{20} \quad \frac{5}{3} a$$

$$\boxed{21} \quad 1,56 \text{ m}$$

$$\boxed{22} \quad 68^\circ$$

$$\boxed{23} \quad 0,13 \text{ }^\circ\text{C}$$

$$\boxed{24} \quad 5 \text{ g}$$

$$\boxed{25} \quad 170 \text{ cm}$$

$$\boxed{26} \quad 0,5 \text{ m/s}$$

$$\boxed{27} \quad \text{A teli víziló } \frac{\sqrt{202}}{14} \text{-szor gyorsabb.}$$

$$\boxed{28} \quad 392 \text{ m. Az elfogadható megoldások tartománya } 385 - 405 \text{ m.}$$

$$\boxed{29} \quad \frac{3\pi r^4}{3\pi r^4 + a^4} \omega$$

$$\boxed{30} \quad \frac{T_0 + T_1}{2}$$

$$\boxed{31} \quad md^2$$

$$\boxed{32} \quad \frac{2}{\sqrt{3}} \frac{mg}{L}$$

$$\boxed{33} \quad \frac{\left(2 + \frac{\pi}{2}\right)}{\left(17 + \frac{\pi}{4}\right)} \doteq 0,2 = 20 \%$$

$$\boxed{34} \quad 364 \text{ km}$$

$$35 \quad T$$

$$36 \quad \frac{g}{2\omega^2}$$

$$37 \quad 1,8 \text{ kN/m}$$

$$38 \quad 1,95 \text{ s}$$

$$39 \quad 354 \text{ lx}$$

$$40 \quad \sqrt{\frac{GM}{R} \frac{2\sqrt{2}}{2+\sqrt{2}}}$$