Dear readers,

in your hands you are holding the booklet of the 27th volume of Náboj Physics. The booklet contains all the physics problems you could have encountered during the competition this year, as well as the solutions to them, from which you can learn a lot. If you have any difficulty understanding any of them, do not hesitate to contact us; we will gladly clarify everything.

This booklet would not exist without the enormous effort of many people who participated in organising Náboj Physics. Most of us are students of Faculty of Mathematics, Physics and Informatics of Comenius University in Bratislava, and some of us also actively participate in organising the Physics Correspondence Seminar (FKS).

Náboj Physics continues with its international tradition as well in the year 2024. For the international cooperation, we would like to thank to the local organisers: Katarína Nedeľková (Bratislava), Marián Kireš (Košice), Jakub Kliment (Prague), Lenka Plachtová (Ostrava), Ágnes Kis-Tóth (Budapest), Urszula Goławska (Gdańsk), Andrzej Karbowski (Toruń), Mirela Kaczmarek (Wrocław), José Francisco Romero García (Madrid) and Dmytro Rzhemovskyi (Vienna).The results of this international clash can be found on our web site.

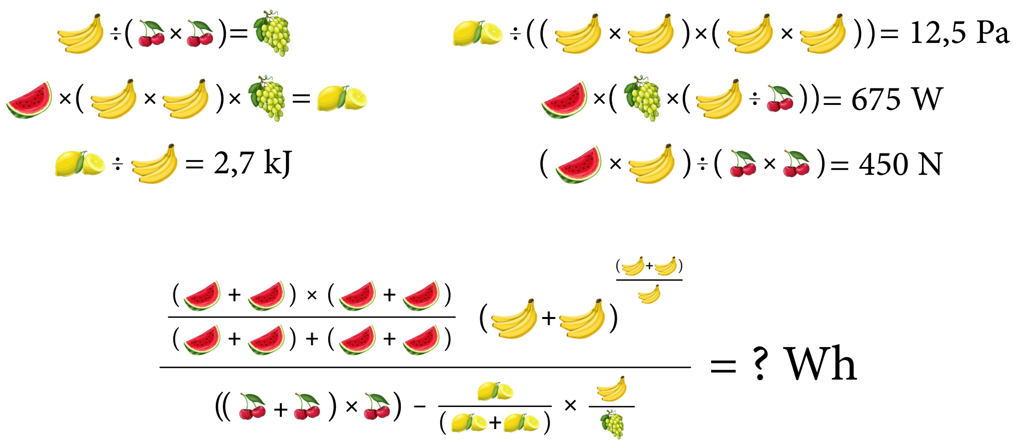
In the name of the entire team of organisers, we believe that you enjoyed Náboj Physics in 2024, and we hope that we see each other at Náboj next year. Either as competitors or organisers.

Jaroslav Valovčan
Chief organiser

The results, the archive and other information can be found on the web site `https://physics.naboj.org/`.

Problems

1 95 % of people cannot solve this problem! Can you?



2 Thomas was watching a car race. This time an American one, which means turning left for three hours. And because he never saved enough money to travel to America to see it live, he called Pato and Josef and asked them to race for him at the local roundabout.

The boys came in their clunkers, stopped next to each other and Thomas started the race. They both drove at a constant breakneck speed of 18 km/h. Pato stayed in the inner lane, a circle with a circumference of 100 m, while Josef drove in the outer lane whose circumference was 120 m. How many times will Pato have circled around the roundabout by the time he overtakes Josef for the first time?

Neglect the time the boys require to reach the breakneck speed.

3 Martin often observes meteor showers. His preparations for all-night observations include, among other things, making coffee in a vacuum flask (seven teaspoons of instant coffee and nine teaspoons of sugar, stirred, not shaken). The flasks's inner container is a cylinder with a height of 18 cm and a base radius of 4 cm. The walls of the inner container are 0.5 mm thick and made of aluminium with density 2.7 $\mathrm{g/cm^3}$ and specific heat capacity 0.9 $\mathrm{J/(g \cdot K)}$. Martin completely filled the inner container with coffee at a temperature of 95 °C.

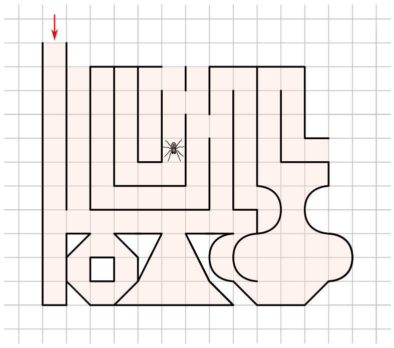
How much will the coffee cool by the time it reaches equilibrium with the inner flask? The inner flask is separated from the outer one by a vacuum, so you can assume that it is perfectly thermally insulated. The specific heat capacity of the coffee is equal to that of water. The original temperature of the flask was 20 °C.

4 Matt has finally boarded the airplane. Now he is just about to take off. As an experienced traveller he is completely indifferent to delays, screaming children and disgusting dry sandwiches, but as a physicist he is still interested in the flight data. So he pulls out his phone, which shows him that the airplane is accelerating at 2 $\mathrm{m/s^2}$. However, the speed of the airplane is usually expressed in km/h.

What is the acceleration of the airplane in the related units of $\mathrm{km/h^2}$?

5 Justine and Martin received a wedding gift: a glass creation made by professional glassmakers in Venice. However, soon a spider crawled deep into the gift. It did not look very aesthetic there, so they decided to flush it out.

How much water do they need to pour into the leftmost pipe, so that the square with the spider is completely filled? Consider that one square corresponds to a volume of 1 l and the dimensions are small enough that the compression of the air can be **neglected**.



6 George measured on the globe that the distance from Rio to Hong Kong is 17 700 km and the distance from Rio to Tokyo is 18 600 km. What is the maximum possible distance between Tokyo and Hong Kong?

The Earth is a ball with a circumference of 40 000 km. *Do not assume anything about the real world locations of the mentioned places.*

7 Two experienced Trackmania racers, Thomas and Matthew, are racing on a 300 m long straight track. Thomas's car starts accelerating straight away from the starting line with a constant acceleration of 8 m/s^2. Matthew's car has ignition issues and only starts one second later, but with an acceleration of 9 m/s^2.

Who will cross the finish line first and by how many seconds will he win?

8 Submit the sum of numbers of all true statements.

1	Water is flowing in a hose through a narrow spot. The velocity of the flow is lower there.
2	If we exert force on the surface of liquid, the pressure is the same at every point inside it.
4	An object is floating on the surface of water in a glass. Its density therefore must be lower than that of water.
8	A narrow glass cylinder is half-full of mercury. When tilted by 45°, the pressure at the bottom decreases.
16	An ice cube is floating on the surface of water in a glass with room temperature. The water level will rise until it melts completely.
32	Water cannot evaporate at temperatures much lower than its boiling point.
64	Two objects are floating on the surface of a body of liquid. The object whose volume above surface of liquid is larger is less dense.

All mentioned objects are at standard room conditions (homogeneous gravitational field, atmospheric pressure, room temperature, etc.)

9 The windows at our local swimming pool are covered by thick curtains. In one of them, there is a small hole, through which a single beam of sunlight hits the surface of water in the pool at an angle of 45°. Since sunlight consists of coloured components, a rainbow strip appears at the bottom of the pool. What is the length of this strip if the refractive index of water for visible light lies within the range of 1.33 – 1.34, and the pool is 2 m deep?

Assume that the refractive index of air is 1.

10 During the long, careless summer days, Marc forgot to pour anti-freeze into his car's windshield wiper fluid reservoir, leaving unspent water there. Now he is standing frozen in front of his car, cursing and gesticulating wildly, and slowly finding out that there is 1 kg of pure ice in the two-litre tank.

He fills it to the top with winter wiper fluid, which freezes at −20 °C. What is the lowest temperature at which the resulting mixture will be liquid, if the density of the antifreeze is 800 kg/m^3 and the freezing point is a simple weighted average based on the mass of the constituents?

11 At the local fair Adam bought a doughnut and a huge apple. He wolfed down the doughnut immediately and saved the apple for the walk home. As he passed over a small bridge, he saw his own reflection in the water. This unsettled him so much that he dropped the apple right into the pond. It turned out that the density of the apple was two thirds of the density of water.

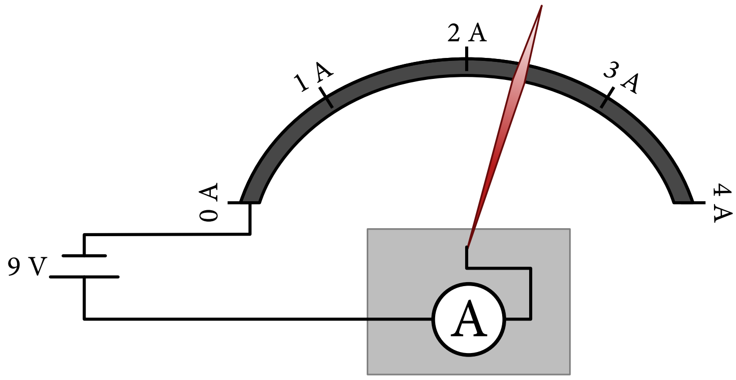
As Adam's hands are still greasy from the doughnut and he does not want to cause an environmental disaster, he knelt over and started chewing on the apple right above the water level. However, the sudden splash also attracted a hungry fish, which started eating the apple below the surface as well.

What part of the apple will Adam have eaten in the end? Given the massive advantage in terms of teeth on Adam's side, he eats three times as fast as the fish.

12 Jeremy is whizzing down the hill on his bicycle. At his current speed and on level ground, he would need a distance s and time t to stop. What is the minimum incline of the hill relative to the horizontal plane that would prevent Jeremy from being able to stop at all?

Jeremy knows how to brake properly and his wheels never skid.

13 Jacob was searching for a multimeter at home, but unfortunately, he only found a weird ammeter. Its needle moves along a thin conductive strip with a length resistance of 4 Ω per segment, just as shown in the picture. He connected this miracle of modern electrical engineering to a battery with a voltage of 9 V. What current will the weird ammeter indicate?



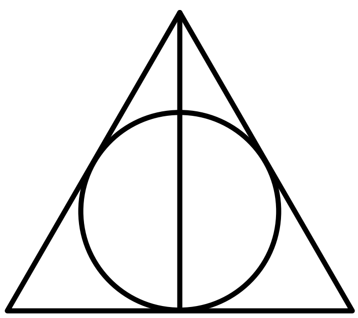
14 Patrick discovered a strange black disc in his grandfather's cabinet. Before he could serve a soy burger on it or use it instead of a frisbee, his grandfather showed him that if he places it on the turntable, positions the needle in the spiral groove at the outer edge of the disc and finally presses a few buttons, ominous crackling and hissing sounds will start coming out from the speakers, along with some prehistoric music.

The hissing and crackling lasted exactly 20 minutes. What is the length of the groove if the outer diameter of the disc is 30 cm and the groove ends 5 cm from its centre? The mysterious disc rotated on the turntable at a speed of 33⅓ revolutions per minute.

15 A jumper dives into a pool from a diving board that is 10 m above the water level. She bounces off the board, performs a somersault and lands in the water. How much time does the jumper spend in the air, if she spent exactly half of the duration of the jump above the level of the diving board?

16 Sarah is a big fan of Harry Potter. She took a pendant of the Deathly Hallows out of her jewellery box and began to admire it. The Invisibility Cloak, the Resurrection Stone, the Elder Wand. And since she is also a curious physicist, she wondered where the centre of mass of this pendant is located.

The pendant consists of an equilateral triangle with a side length of a, a circle inscribed in the triangle, and one of its heights. All these components have the same linear density. How far is the centre of mass of the pendant from the top vertex of the triangle?



17 Andrew built a primitive catapult. It consists of a massless plank with length $2L$ with a hinge at its centre, that is attached to a crude frame at height L above the ground. He glued a heavy stone counterweight with mass M to one end and put a tiny projectile with mass $m < M$ into a shallow pit on the other end.

The projectile lies there until Andrew pulls the latch and fires the catapult. The heavy stone descends, and the projectile leaves the pit, right when the plank is vertical.

How far from the counterweight will the projectile impact the ground?

18 The highway from Bratislava to Prague is exactly 400 kilometers long and has four lanes in each direction. In those lanes, cars travel at speeds of 80 km/h, 100 km/h, 120 km/h and 160 km/h. Peter, Paul, and Arthur left Bratislava at the same time, but each took a different lane because each car can handle different speeds. When they arrived in Prague, they bragged about their experiences:

Peter: "I drove in the fastest lane and overtook 620 cars on the way, and out of those, 200 were in the second fastest lane!"

Paul: "I was driving at a speed of 120 kilometers per hour and overtook a total of 220 cars!"

Arthur was driving at a speed of 100 km/h, but doesn't remember how many cars he overtook. Calculate it for him!

Assume that cars depart from Bratislava in each lane uniformly and all travel the entire distance.

19 Matt likes to horse around with his physicist friends. Today they realized that if

- one *horse length* is eight feet;
- one *horsepower* is the power required to lift a mass of 75 kg straight up at a speed of 1 m/s;
- and one *horse ration* is 1.7 kg of feed per 100 kg of a horse's weight per day;

it is possible to construct a system of measurement in which they can express any mechanical quantity. For instance, *horse acceleration* would be one *horse length* multiplied by one *horse ration* squared.

How many kilograms is one *horse weight*, that is, a large unit of mass derived from these three units?

20 Cathy and Billy are riding an escalator. It has a length of 120 visible steps and moves at a speed of two steps per second. They both step on the lowest step. Cathy, as a responsible adult, stands there calmly, while Billy starts running frantically back and forth between her and the top of the escalator, at a speed of two steps per second on his way up and six steps per second on his way down.

How many steps in total will he have run over by the time Cathy catches him at the top of the escalator and gives him a good tongue-lashing?

21 Marc and Sabine are flying in two supersonic jet aircraft, each at a speed of Mach 3, along two parallel lines 1 km apart in opposite directions. How much time will elapse between their closest approach and the instant when Sabine hears the sound of Marc's aircraft?

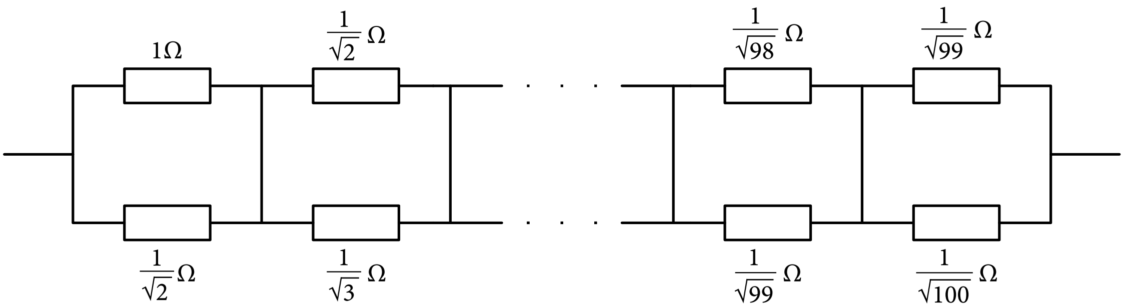
Speed of sound is Mach 1 = 1 km *in* 3 s.

22 Marcel and Sabine are going on a honeymoon in a hot air balloon. However, their budget is already quite tight, and even though Marcel somehow managed to borrow a balloon for free, he still needs to fill it with something. Helium is expensive, so they will have to settle for electrolytically obtained hydrogen.

How much will it cost them to fill the balloon, if the empty balloon with the newlyweds has a mass of 1000 kg, the price of electricity for young couples is 0.20 €/kWh, and the overall efficiency of the process is 50 %? Combustion of hydrogen with oxygen produces 285.8 kJ/mol of energy.

The honeymoon will be taking place at standard temperature and pressure. The pressure inside the balloon is equal to the outside pressure.

23 Sam found a huge box of various resistors and a long coil of perfectly conductive wire and built a long ladder from them. Between each pair of rungs, there is a resistor with resistance $\frac{1}{\sqrt{n}}\,\Omega$ on one side and a resistor with a resistance of $\frac{1}{\sqrt{n+1}}\,\Omega$ on the other side, for n ranging from 1 to 99.



What is the total resistance between the ends of the ladder?

24 The height of the sea level influenced by the tide in the French city of Saint-Malo around the full moon can be modelled as $A \cdot \sin(\omega t)$, where A is the amplitude and ω is the frequency such that the tide occurs every 12 hours.

The sailors of the ship Santiano would be interested to know if they will be able to dock when they come home. The problem is that the ship can only anchor at the pier if the tide is at most $A/3$ lower than at high tide… and the sailors do not know at what time the high tide occurs.

What is the probability that they will be able to anchor at the pier, that is, they arrive at the time when they can anchor?

25 Max and Moritz measure distances between places on the Earth in different ways: Max, a pilot, gives the distance as the length of the shortest curve along the surface of the planet; while Moritz, a bank robber, gives the distance as a straight line, even if it goes under the surface.

When they measured the distance between the airport and the bank, they found that the difference in the lengths they measured was exactly 1 m. What distance did Max measure?

The Earth is a ball with radius of 6371 km.

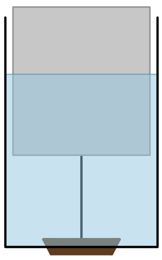
26 Bob the bob is swinging on a swing suspended by chains of length L. In the extreme position, when he comes to a stop for a brief moment, he feels a g-force of $0.5\ g$. What g-force does Bob feel at the lowest position?

27 Kate got a new metal necklace: it is in the shape of a circular loop with a radius r and linear resistance λ. During her vacation, she went for a ride on a merry-go-round at the local playground. She sat at a distance R from the axis of rotation and started spinning around it at an angular velocity ω. At that moment, she noticed that some rascals had placed a huge magnet under the merry-go-round, creating a magnetic field with induction B that is uniform in size and directed along the axis of rotation.

Kate began to worry about her necklace: what current is induced in it? Assume that Kate is frozen stiff with fear and remains in one place. She is sitting in such a position that the necklace makes an angle of 30° with the horizontal plane.

28 Lucy has devised an ingenious device for proper dosing of water to her plants. The dispenser consists of a sufficiently tall cylindrical container with a base area S, which has a circular opening in the centre of its bottom extending halfway through its radius, covered by a massless plug. Inside the container, there is a smaller cylinder with a density of $500\ \mathrm{kg/m^3}$, a base area of $0.99S$ and a height of H, which is attached to the plug at the bottom by a cord. After filling the vessel with a certain critical amount of water, the inner cylinder lifts the plug from the bottom, preventing any further water from being added to the container.

What is the smallest and the largest volume of water that Lucy can dose with this measuring device, if she can set the length of the cord to any desired length? The volume of the inner cylinder is 1 l.



29 Dan's office is a terrible mess. The one thing that might strike your eye when you walk is his prized toy: a pair of pendula.

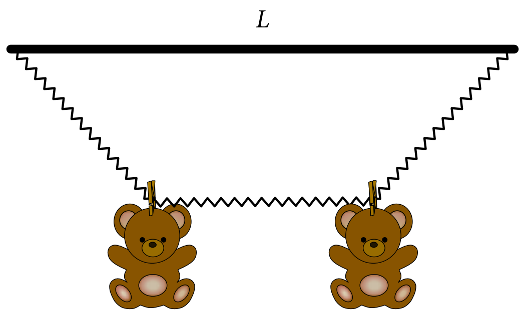
When they are simultaneously displaced from their equilibrium position, their motion has the following characteristics:

- exactly with every third passage of the slower pendulum through the extreme position on the right, it meets the faster one there, and
- exactly with every fifth passage of the faster pendulum through the extreme position on the left, it meets the slower one there.

The length of the string of the longer pendulum is 10 cm. What is the length of the string of the shorter pendulum?

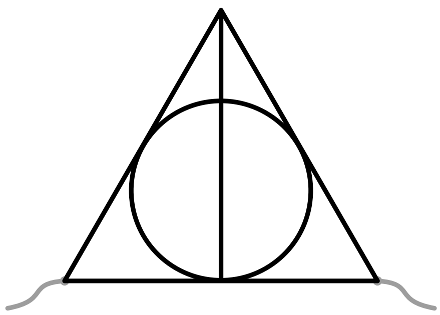
30 Penny needs to dry two freshly washed plushies, each with a mass of m. Unfortunately, during her first attempt to hang them on the clothesline, it snapped. But Penny was too clever for her own good – she stretched a spring with zero resting length and stiffness k over the now empty horizontal frame of the dryer with width L. Then she hung the plush toys at one third and two thirds of the length of the spring, which sagged under their weight.

How much lower will the centre of the spring be relative to its ends?



31 Sarah once again pulls out her favorite Deathly Hallows pendant. This time she is interested in its electrical resistance. Sarah's pendant consists of an equilateral triangle with a side length of a, an inscribed circle inside it, and one of its heights. Linear resistance of all its components is λ.

What is the resistance between the two vertices of the triangle that are not touched by the wand?



32 The Dark Lord™ has spent some quality time in the void, and now he is returning to the world of mortals with a new black magic spell: turning off the gravity. At this moment he can only suspend it for a short while, but surely he must be planning something much bigger.

Now he is asking you (out of purely academic interest, of course): for how long would he need to turn off the Sun's gravity so that the Earth would fly away to infinity?

33 Ellie took an old double hotplate for her cooking experiments. Despite her expectations, the hotplate worked very well and after she plugged it in, it warmed up to an astronomical temperature of 250 °C. However, at such temperatures, the food would char easily. Unfortunately, the controls on the old hotplate did not work anymore, so Ellie had to improvise. She took a sprayer with a special cooling liquid at a temperature of 20 °C and applied it to the heated hotplate with a constant volumetric flow rate. What must the volumetric flow rate be so that the temperature drops to acceptable value of 150 °C? The surface area of the cooking plate is 0.1 m^2, and after years of heavy use it is absolutely and irrevocably black.

The special liquid has a constant specific heat capacity and density over a wide range of temperatures, which are equal to the specific heat capacity and density of water at room temperature. It also boils at a temperature of 100 °C*, just like water, and has the same specific latent heat of vaporization.*

34 Jacob is carrying precious cargo in his car – a wedding cake in the shape of a rotational paraboloid with height H a base with radius R. As he approaches an intersection, the traffic light suddenly turns red, forcing him to slam on the brakes. What is the greatest deceleration he can apply so that the cake will not topple over?

Ignore effects caused by jerk.

35 After a long and bright life, our Sun will die. The process of dying looks like this: it will shed a layer of material from its surface into space under the influence of radial forces.

Consider that the Sun is a sphere with a rotational period of 28 days, consisting of a homogeneous core and an outer homogeneous envelope. The mass ratio of the core to the envelope is 1 : 1 and the ratio of their densities is 63 : 1. When the Sun dies, it will shed its entire envelope, and the core will collapse into a white dwarf – a homogeneous sphere with a radius of 5000 km.

What will be the rotational period of the white dwarf?

36 The flux of solar neutrinos through the Earth is about $10^{15}\ \mathrm{m^{-2} \cdot s^{-1}}$. Dirk built a huge detector in the shape of a cube with a volume of 1000 m^3, filled it with water, and found that on average, one neutrino is captured in it every second.

What is the mean free path of a neutrino in water?

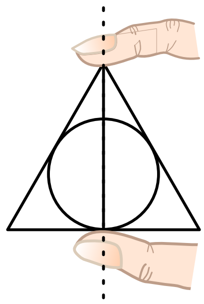
37 After the Olympics, we found ourselves fascinated by pistol shooting and built an airgun. The mechanism of an airgun consists of a cylinder with a volume of $V = 100\ \mathrm{cm^3}$ filled with air at atmospheric pressure and closed by a piston with a mass of $M = 1$ g. Before the shot, the shooting mechanism is compressed to a volume of $\frac{V}{16}$. Subsequently, a projectile with a mass of $m = 2$ g is placed in front of the piston. When the trigger is pulled, the piston is released and the air expands, propelling the projectile.

To test the gun, we hired two shooters, Yusuf and Kim. Yusuf arrived at the sport, loaded the gun, pulled the trigger and left. When the second tester, Kim, arrived, we had plenty of time: between loading and shooting, she first adjusted a flap over her left eye, so that she saw the target only with her right one, and then she placed an aperture with a small hole in front of her right eye, allowing her to see the target very sharply. While she was preparing, the air in the piston cooled down. Now we are interested in the ratio of the velocities of Yusuf's and Kim's fired projectiles.

The work done by the gas during an adiabatic process is $W = \frac{p_0 V_0 - p_1 V_1}{\kappa - 1}$*, where index* 0 *represents the initial state and* 1 *the final state. Air is an ideal diatomic gas with* $\kappa = 1.4$.

38 Sarah picked up her Deathly Hallows pendant for the third time. She held it between her thumb and index finger, so that the elder wand lay directly on the line connecting these two fingers, and then she spun it. What was the moment of inertia of the pendant with respect to that axis?

Sarah's pendant geometrically consists of an equilateral triangle with a side length of a, an inscribed circle, and one of its heights. The linear density of all its components is λ.

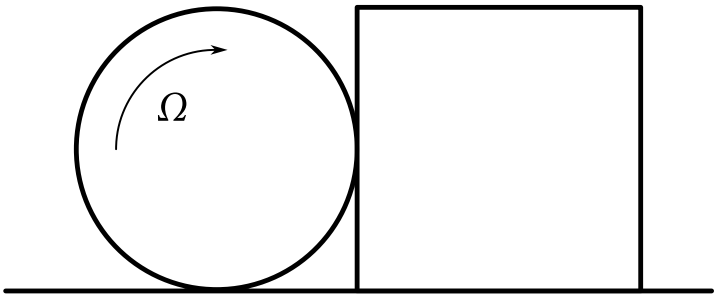


39 On her birthday, Grandma Justine has to hear Happy Birthday from her extended family. Every year she needs to hear it at a sound intensity level (measured in decibels) equal to her age in years, and we had to try harder and harder. Last year we were not able to sing any louder, so we enlisted the help of our noisy neighbour, who is capable of singing with a sound intensity level of 100 dB at a distance of 1 m.

This year we found out that even our neighbour's help wouldn't be enough. Instead, we moved Grandma Julie's armchair 30 cm closer to the singing relatives. How old is Granny Julie this year?

40 Danny the cube and Johny the cylinder are fighting once again. With no fists or legs, the fight proceeds as follows: Johny spins up to an angular speed $\Omega = 23\ \mathrm{s}^{-1}$, lies down right next to Danny and starts pushing into his side. Cubical Danny is fairly indifferent to being pushed around, but he is interested what maximum speed he will achieve while Johny is pushing him.

The coefficient of friction between Danny and the ground and between Danny and Johny is $f = 0.2$, and the coefficient of friction between Johny and the ground is twice as large. All Danny's edges, Johny's height and diameter are 1 m and they both weigh 100 kg.



Solutions

1 We hope you have passed the hidden IQ test and denoted each fruit with some letter. For example B – banana, L – lemon, G – grape, M – watermelon and C – cherry. The gargantuan equation is then simplified to

$$\frac{M(2B)^2}{2C^2 - \frac{1}{2}\frac{B}{G}} = ?\ \mathrm{Wh}. \tag{1.1}$$

We plug the equation $\frac{B}{C^2} = G$ in the denominator. Next, we multiply the whole expression by $\frac{BL}{BL}$, which leads to

$$\frac{M(2B)^2}{2C^2 - \frac{1}{2}\frac{B}{G}} = \frac{4MB^2}{\frac{3B}{2G}} = \frac{8}{3}MBG = \frac{8}{3}\frac{L}{B}\frac{MB^2G}{L}. \tag{1.2}$$

We notice that the last fraction is equal to unity, as can be deduced from one of the equations from the problem statement, and using another equation, $\frac{L}{B} = 2.7\ \mathrm{kJ}$. That means that fraction of interest is equal to 7.2 kJ. In order to convert it to Wh, we recall that 1 Wh = 3600 J, so the result is 2 Wh.

We firmly believe that from now on you will not complain about letters in equations.

2 Both racers drive at constant speed and therefore at any time they will have covered the same distance. If Pato drives n times around the roundabout, he will have travelled a distance of $n \cdot 100$ m. We require Josef to drive one circle less, so at the same time he has driven the distance of $(n-1) \cdot 120$ m. Since the covered distances must be equal, we know that

$$n \cdot 100\ \mathrm{m} = (n-1) \cdot 120\ \mathrm{m}, \tag{2.1}$$

which holds for $n = 6$.

3 What happens with Martin's coffee? It will start giving away heat to the aluminium container which begins to heat up. The heat transfer stops once both coffee and container have the same temperature, and the heat given away by coffee equals the heat received by the container.

The volume of the inner container is

$$V = \pi r^2 h \doteq 905\ \mathrm{ml}, \tag{3.1}$$

where r is the base radius and h is the container's height. The coffee with density of water therefore weighs $m_k = V\rho_{\mathrm{H_2O}} \doteq 905$ g, its specific heat capacity is $c_{\mathrm{H_2O}}$ and its temperature went down from T_k to some lower temperature T.

Then there is the container – a cylinder with mass

$$m_{\mathrm{Al}} = \left(2\pi r^2 + 2\pi r h\right) \Delta h \rho_{\mathrm{Al}} \doteq 75\ \mathrm{g}, \tag{3.2}$$

where Δh is the wall thickness and ρ_{Al} is the aluminium density. The container with specific heat capacity c_{Al} had its temperature increased from initial T_{Al}, again to temperature T. The calorimetric equation therefore reads

$$m_k c_{H_2O}(T_k - T) = m_{Al} c_{Al}(T - T_{Al}), \tag{3.3}$$

whence

$$T = \frac{m_k c_{H_2O} T_k + m_{Al} c_{Al} T_{Al}}{m_k c_{H_2O} + m_{Al} c_{Al}}. \tag{3.4}$$

The coffee will therefore cool by $T_k - T \doteq 1.3\ °C$.

4 If we want to convert units, for example from metres to kilometres, we usually multiply the value by number one in a suitable form,

$$1 = \frac{1 \text{ new unit}}{x \text{ old units}}. \tag{4.1}$$

For example we can convert 563 m to kilometres by writing

$$563 \text{ m} = 563 \text{ m} \cdot \underbrace{\frac{1 \text{ km}}{1000 \text{ m}}}_{1} = \frac{563}{1000} \text{ km} \cdot \frac{\cancel{m}}{\cancel{m}} = 0.563 \text{ km}. \tag{4.2}$$

Equivalently we convert m/s^2 to km/h^2. Since 1 h = 3600 s, suitable ones are $\frac{1 \text{ km}}{1000 \text{ m}}$ and $\frac{3600 \text{ s}}{1 \text{ h}}$.

The entire computation is as follows:

$$2 \text{ m/s}^2 = 2 \text{ m/s}^2 \cdot \frac{1 \text{ km}}{1000 \text{ m}} \cdot \left(\frac{3600 \text{ s}}{1 \text{ h}}\right)^2 = \frac{2 \cdot 3600^2}{1000} \cdot \frac{\cancel{m}}{\cancel{m}} \cdot \frac{\cancel{s^2}}{\cancel{s^2}} \cdot \text{km/h}^2 = 25\,920 \text{ km/h}^2. \tag{4.3}$$

5 As water flows into the unusual glass creation, it follows certain rules:

1. If it can flow somewhere lower, it will.
2. Therefore the free surface must be at the same height everywhere.
3. At the beginning, there is air everywhere. So if water is to flow somewhere, there must be an opening for the air to escape. If there is no opening, the air will prevent the next water from flowing in.

Gradually, as we pour in water and adhere to these rules, we will reach a filling level as shown in the image 5.1.

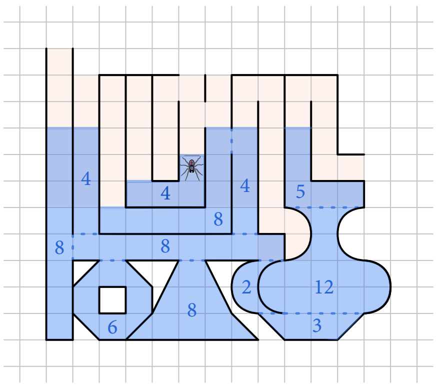


Figure 5.1: *Distribution of water in the container*

All that remains is to calculate the volume of the poured water. If we realize that the cut parts have a volume of 1 l or 0.5 l and the arc-shaped parts complement each other to form whole squares, the volume of the poured water can simply be determined as 72 l.

6 Let us find a solution on a flat Earth first. We draw two circles around Rio, with radii of 17 700 km and 18 600 km, and then try to find a pair of point whose relative distance is largest. These are, quite trivially, any two points on the opposite sides on from Rio and their relative distance is the sum of their radii, or 36 300 km.

On a real spherical Earth it looks similar, however no two points can be further apart than half of its circumference, or 20 000 km – otherwise it would be shorter to turn around and take the other route. The shorter distance is then the complement to the circumference, which is only 40 000 km − 36 300 km = 3700 km.

Alternatively, we may realize that all points at a distance of x from Rio are at a distance of $\pi R - x$ from the antipodal point to Rio (exactly on the other side of the planet). This means they lie on a circle with this radius, and we need to find a pair of most distant points on two circles with radii 1400 km and 2300 km, which is again 3700 km.

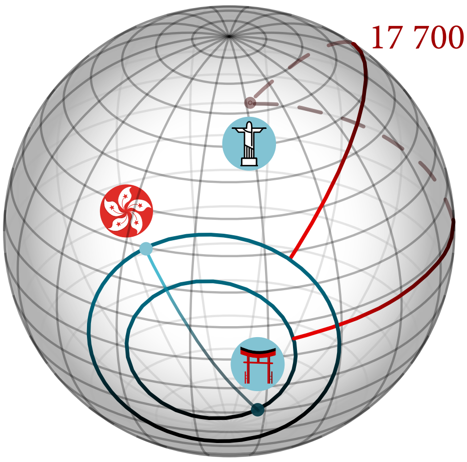


Figure 6.1: *Two possible places where Tokyo (torii) and Hong Kong (flower) might be located. Rio (the statue of Christ) lies on the other side of the Earth.*

7 Let us denote the length of the $L = 300$ m and Thomas's and Matthew's acceleration $a_\mathrm{T} = 8$ m/s and $a_\mathrm{M} = 9$ m/s. Time t_T which it took Thomas to reach the finish line, can be calculated from the equation for motion with constant acceleration

$$L = \frac{1}{2} a_\mathrm{T} t_T^2, \tag{7.1}$$

from where we can express

$$t_\mathrm{T} = \sqrt{\frac{2L}{a_\mathrm{T}}}. \tag{7.2}$$

Similarly, we calculate the time t_M which Matthew took to cross the finish line as

$$t_M = \sqrt{\frac{2L}{a_M}} \tag{7.3}$$

We add $\Delta t = 1$ s to this time and we compare the total times both racers required to reach the finish. After plugging in the values from problem statement we get

$$(t_M + \Delta t) - t_T = 0.505 \text{ s}, \tag{7.4}$$

so Thomas wins by 0.505 s.

8 The numbers are powers of two, so we must decide the truthfulness of all statements.

Hose (1)

The equation of continuity applies: what flows into something must flow out as well. The volumetric flow is constant, and thus a smaller cross-section means a greater velocity. The statement is **false**.

Force acting on a surface of a closed container (2)

Pascal's law could suggest something similar to us. But in addition to the pressure caused by our force, there is also the weight of the liquid, and the pressure it causes is called hydrostatic pressure – this, however, varies with depth, so the statement is **false**.

Body on the surface (4)

The statement is **false**, whether we recall a needle floating on the water surface due to surface tension or a floating iron ship.

Mercury in a measuring cylinder (8)

The pressure at the bottom of the cylinder is hydrostatic pressure, which depends on how deep below the surface this bottom is. By tilting the cylinder, its horizontal cross-section increases, and therefore the liquid level decreases. The hydrostatic pressure thus decreases as well, and the statement is **true**.

Floating ice cube (16)

The submerged part of the ice cube occupies the same volume as water with the same weight – therefore when the cube melts, the newly formed water only occupies the space that the ice cube occupied below the surface. The claim is **false**.[1]

Evaporation (32)

The statement is **false** – even at a lower temperature, the molecules with the highest energy can escape from the surface of the liquid. When boiling point is reached, the liquid evaporates in its entire volume.

[1] If we consider that the temperature of the water decreased due to the melting ice and consequently its volume due to thermal expansion, we would conclude that the level would even drop.

Two bodies (64)

Even though polystyrene foam has a much lower density than wood, a large log and one polystyrene ball can float next to each other – and this is in direct contradiction to the statement in the problem, so it is **false**.

Summing the numbers of the true statements we get the answer 8.

9 When a light ray passes through an optical interface, it refracts according Snell's law. The considered ray passes from a material with refractive index 1 (air) to a material with refractive index n (water) at the angle of incidence 45°. Therefore

$$\frac{\sin 45^\circ}{\sin \alpha} = n. \tag{9.1}$$

If we consider a right triangle determined by the ray, normal to the interface at the place of incidence, and bottom of the pool, according to the Pythagorean theorem, the path length of the ray in the water is $\sqrt{d^2 + h^2}$, where h is depth of the pool and d is distance of the place at the pool bottom which is hit by the ray from the normal. Thus we can write

$$\sin \alpha = \frac{d}{\sqrt{d^2 + h^2}}, \tag{9.2}$$

After substituting into the previous equation, we can express

$$d = \frac{h}{\sqrt{2n^2 - 1}}. \tag{9.3}$$

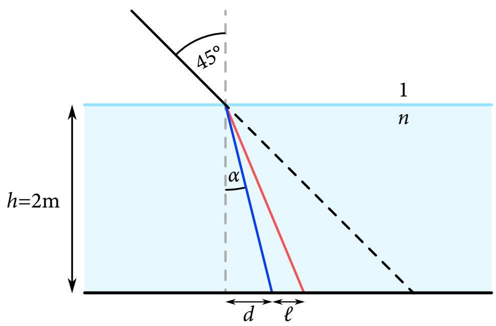


Figure 9.1: *The outermost rays hitting the pool bottom*

Individual colour components of light differ (among other things) by their refractive index. Therefore, each wavelength refracts slightly differently after hitting the interface. From the problem statement we know that the refractive index of water for visible wavelengths lies within the given range – let's denote the limit values n_{min} and n_{max}. Each of them leads to a different value of d. A difference of the ds is the sought length of the rainbow stripe at the bottom of the pool

$$\ell = \left| \frac{h}{\sqrt{2n_{\text{min}}^2 - 1}} - \frac{h}{\sqrt{2n_{\text{max}}^2 - 1}} \right|. \tag{9.4}$$

For the provided numerical values $\ell \approx 13$ mm.

10 In the tank, there is $m_\mathrm{i} = 1\,\mathrm{kg}$ of ice with density of $\rho_\mathrm{i} = 916\,\mathrm{kg/m^3}$, therefore its volume is $V_\mathrm{i} = m_\mathrm{i}/\rho_\mathrm{i} \doteq 1.092\,\mathrm{l}$.

If we denote the total volume of the tank as V, the antifreeze mixture occupies a volume of $V_\mathrm{a} = V - V_\mathrm{i} \doteq 0.908\,\mathrm{l}$, and therefore its mass is $m_\mathrm{a} = (V - V_\mathrm{i})\rho_\mathrm{a} \doteq 0.727\,\mathrm{kg}$. The new freezing temperature is a weighted average according to the mass of the constituents, which is

$$T = \frac{m_\mathrm{i} T_\mathrm{i} + m_\mathrm{a} T_\mathrm{a}}{m_\mathrm{i} + m_\mathrm{a}}, \tag{10.1}$$

where T_i and T_a are the freezing temperatures of water and the antifreeze mixture.

For the values from the problem statement it equals $T \doteq -8.4\,°\mathrm{C}$.

11 Whenever Adam or the fish bite off a piece of the apple, due to buoyancy it will find a new position, such that the submerged part of the apple will always constitute two thirds of its volume.

Therefore at any given time, both of them have something to bite, however small the remaining piece of the apple is. And because both of them are eating all the time and finish simultaneously, the ratio of eaten parts will be determined by the ratio of their biting rates. Therefore, the answer is $\frac{3}{4}$.

12 Suppose Jeremy's speed is v_0. While braking on flat ground, he decelerates with constant deceleration a, so his speed $v(t)$ changes in time as

$$v(t) = v_0 - at. \tag{12.1}$$

We see that $v(t) = 0$ when $t = v_0/a$, which is the time required to come to a halt. The travelled distance is then

$$s = v_0 t - \frac{1}{2}at^2, \tag{12.2}$$

which means that after time $t = v_0/a$, so before fully stopping, Jeremy has covered the distance

$$s = \frac{1}{2}at^2, \tag{12.3}$$

from where we easily express Jeremy's deceleration

$$a = \frac{2s}{t^2}. \tag{12.4}$$

Using simple trigonometry, we know that if Jeremy is travelling down an inclined plane at an angle α with respect to the horizontal plane, his acceleration is $g \sin\alpha$. If this acceleration is the same as Jeremy's deceleration, he will never stop, so the angle α is

$$\frac{2s}{t^2} = g\sin\alpha \quad \Rightarrow \quad \alpha = \arcsin\frac{2s}{gt^2}. \tag{12.5}$$

13 Even if the circuit may appear strange, Ohm's law is still a good tool to use. In this specific case, the resistance in the circuit depends on the current showed by the ammeter. Specifically, if the ammeter shows there is a current of k amperes, the resistance will be $4k\ \Omega$.

From the Ohm's law $U = RI$ we obtain

$$9\ \mathrm{V} = 4k\ \Omega \cdot k\ \mathrm{A}, \tag{13.1}$$

and from that,

$$k = \frac{3}{2}. \tag{13.2}$$

The ammeter will indicate $\frac{3}{2}$ A.

14 Since we know the angular speed of the LP record[2], we can determine that after 20 minutes it completed $20 \cdot 33⅓ = 666⅔$ revolutions. In this time, the needle moves along the entire groove, so the groove must twist around the centre of the disc 666⅔ times. Since the diameter of the disc is 30 cm and the groove ends 5 cm from the centre, the grooved part of the disc is 15 cm − 5 cm = 10 cm wide, so the distance between two twists of the groove is approximately

$$w \approx \frac{10\ \mathrm{cm}}{666⅔} = \frac{3}{20}\ \mathrm{mm} = 0.15\ \mathrm{mm}. \tag{14.1}$$

Area S covered by the groove is easily calculated as the difference of the surface of whole disc and the ungrooved part around the centre,

$$S \approx \pi\left((15\ \mathrm{cm})^2 - (5\ \mathrm{cm})^2\right) = 200\pi\ \mathrm{cm}^2. \tag{14.2}$$

Finally, we untwist the groove into the shape of a very long rectangle. If we know its surface and width, the length ℓ is equal to their quotient,

$$\ell \approx \frac{S}{w} = \frac{200\pi\ \mathrm{cm}^2}{0.15\ \mathrm{mm}} \approx 418.88\ \mathrm{m} \doteq 419\ \mathrm{m}. \tag{14.3}$$

15 The speed of the jumper during her jump can be decomposed into horizontal and vertical components. The horizontal component is constant and not very interesting to us. Thus, we only need to model the entire situation as a vertical throw. Let's denote the initial speed of the jumper as v_0. This speed changes linearly throughout the motion and this rate of change is the gravitational acceleration g.

At the beginning, the jumper had a speed of v_0, and at the moment when she passed the level of the diving board downwards, her speed was $-v_0$. The difference in this two speeds is therefore $2v_0$ and the time the jumper spent above the level of the diving board must be $\frac{2v_0}{g}$. This is to be half of the total flight time T, thus

$$T = \frac{4v_0}{g}. \tag{15.1}$$

Now let's express the initial speed v_0 as a function of the height of the diving board, $h = 10$ m. We start from the moment when the jumper passes the level of the diving board downwards For her uniformly accelerated motion downwards we can express h as

$$h = v_0\frac{T}{2} + \frac{1}{2}g\left(\frac{T}{2}\right)^2. \tag{15.2}$$

[2] We hope that Patrick's disc was indeed an LP record.

Finally we express v_0 from equation 15.1 and substitute it into equation 15.2. This simplifies to

$$h = \frac{gT^2}{4}, \tag{15.3}$$

from which we can easily express

$$T = 2\sqrt{\frac{h}{g}} \approx 2\ \mathrm{s}. \tag{15.4}$$

16 First, let's consider what the geometry of the necklace looks like. The vertical line segment (the Elder Wand) forms the median, the height, and also the angle bisector of the equilateral triangle (the Invisibility Cloak). We can calculate its length from the Pythagorean theorem as $\frac{a\sqrt{3}}{2}$. Since it is an equilateral triangle, all other medians will also be the angle bisectors. Thanks to this, we know that the centroid of the triangle will coincide with the centre of the inscribed circle (the Resurrection Stone). Its radius will also be one-third the length of the median, i.e. $\frac{a}{2\sqrt{3}}$.

From the symmetry of the pendant we know that the centroid will be located on the line segment. We can then calculate the position of the centroid as the average of the distances of the individual centroids weighted by their masses. Thus, we only need to calculate the vertical position of the centroid. For simplicity, we can choose the point from which we measure the distance to the centroid as the origin. Since mass is directly proportional to the length of the wire, the distance of the centroid from the vertex of the triangle will be

$$x = \frac{m_\triangle x_\triangle + m_\circ x_\circ + m_| x_|}{m_\triangle + m_\circ + m_|} = \frac{3a\frac{a}{\sqrt{3}} + \frac{\pi a}{\sqrt{3}}\frac{a}{\sqrt{3}} + \frac{a\sqrt{3}}{2}\frac{a\sqrt{3}}{4}}{3a + \frac{\pi a}{\sqrt{3}} + \frac{a\sqrt{3}}{2}} = \frac{6 + \frac{2\sqrt{3}\pi}{3} + \frac{3\sqrt{3}}{4}}{6\sqrt{3} + 2\pi + 3} a \doteq 0.555a. \tag{16.1}$$

17 The projectile's launch speed is determined by the law of conservation of energy. At the beginning, both bodies are located at a height of L above the ground and are stationary. Their total mechanical energy E_0 is thus composed only of potential energy, which we can choose as we please, for instance with respect to the ground,

$$E_0 = mgL + MgL. \tag{17.1}$$

At the moment when the pole is vertical, both bodies are moving with a speed of v, but the counterweight is on the ground, and the projectile is at a height of $2L$, thus the total mechanical energy is

$$E_1 = mg2L + \frac{1}{2}Mv^2 + \frac{1}{2}mv^2. \tag{17.2}$$

Since mechanical energy is conserved, it holds that $E_0 = E_1$, i.e.,

$$MgL + mgL = mg2L + \frac{1}{2}Mv^2 + \frac{1}{2}mv^2, \tag{17.3}$$

from which we can express the launch speed of the projectile as

$$v = \sqrt{2gL\frac{M-m}{M+m}}. \tag{17.4}$$

This speed has a horizontal direction.

From now on, the projectile moves along a parabola, since in the vertical direction it is accelerated by gravity. We want to know how long it takes to traverse a length of $2L$ to fall to the ground. This will simply be

$$\frac{1}{2}gt^2 = 2L \quad \Rightarrow \quad t = \sqrt{\frac{4L}{g}}. \tag{17.5}$$

In the horizontal direction, the projectile moves at a constant speed v and in time t it travels to the distance of

$$s = vt = 2\sqrt{2\frac{M-m}{M+m}}L. \tag{17.6}$$

18 Let's assume that all the drivers departed from Bratislava at 10:00 – for the sake of simplicity.

We shall denote the lanes A, B, C and D from the slowest to the fastest. Numbers of cars leaving Bratislava in the specific lanes per minute are a, b, c and d. At first, we need to calculate how long the journey in every lane takes: it is respectively 5, 4, 3⅓ and 2.5 hours.

Arthur in lane B arrived in Prague at 14:00. Meanwhile, some cars were arriving in Prague in the slowest (A) lane – these left Bratislava at 09:00. That does mean Arthur overtook all the cars in lane A which had departed from Bratislava between 09:00 and 10:00 – that is $60a$ cars.

How many cars were overtaken by Peter, the fastest driver? Arriving in Prague at 12:30, he had to overtake the cars in lane C, which had left Bratislava between 9:10 and 10:00, that means $50c$ cars. And the problem states this number is 200, therefore $50c = 200$.

If we make similar calculations for both the drivers and all the lanes, we find that the problem can be reduced to this simple series of equations:

$$\begin{aligned} 50c &= 200, \\ 150a + 90b + 50c &= 620, \\ 100a + 40b &= 220. \end{aligned} \tag{18.1}$$

From that, we can isolate $a = 1$, and therefore $60a = 60$ and Arthur overtook 60 cars on his way.

19 Two of the units, *horse length* and *horse ration*, can be transformed to SI directly, as their dimensions are metres and reciprocal seconds respectively:

- *horse length* ζ is 8 ft ≐ 2.438 m and
- *horse ration* δ is $\frac{1.7\ \mathrm{kg}}{100\ \mathrm{kg}\cdot 1\ \mathrm{d}}$, or about $1.97 \cdot 10^{-7}\ \mathrm{s}^{-1}$.

A *horsepower* is, as its name hints, a unit of power, and its dimension must be the same as that of a watt. When we plug in the values into its definition, we find out that one *horsepower* ψ corresponds to 735.5 W[3]. In base units, the dimension of a watt is

$$\mathrm{W} = \mathrm{J/s} = \mathrm{N}\cdot\mathrm{m/s} = \mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}^3. \tag{19.1}$$

[3] Or 750 W if we use $g = 10\ \mathrm{m/s^2}$.

To obtain a unit of mass, we need to divide power by a squared unit of length and a cubed unit of reciprocal time. Then

$$1\ \text{horse weight} = \frac{\psi}{\zeta^2\delta^3} \doteq \frac{735.5\ \text{kg}\cdot\text{m}^2\cdot\text{s}^{-3}}{(2.438\ \text{m})^2\cdot(1.97\cdot10^{-7}\ \text{s}^{-1})^3} \doteq 1.624\cdot10^{22}\ \text{kg}. \tag{19.2}$$

20 Regardless of what Billy's motion looks like, two conditions must always hold:

- the total time of his run must be equal to the time Cathy spends on the escalator;
- he must end at the same position as Cathy, so the total number of stairs he had run up and down must be equal.

Since his speed while running up is only one third as fast as when running down, the ratios of total time spent running up and down must be reciprocal, or 6 : 2 = 45 : 15. Since Cathy will obviously spend 60 seconds standing on the stairs, Billy will be running up for a total of 45 s and down for 15 s.

The total distance he will have run will thus be

$$45\ \text{s}\cdot 2\ \text{stairs/s} + 15\ \text{s}\cdot 6\ \text{stairs/s} = 180\ \text{stairs}. \tag{20.1}$$

21 At any moment, sound spreads from the moving aircraft in all directions at the speed of sound c with respect to the atmosphere. Since the airplane is moving at a speed v greater than c, the envelope of the generated sound waves (shock wave) has the shape of a cone in the direction of the airplane. with a vertex angle of 2α, for which it holds that $\sin\alpha = \frac{c}{v} = \frac{1}{3}$. The vertex of this cone moves in the direction of the aircraft (together with the source) at the speed v, and its surface spreads in the perpendicular direction at speed c.

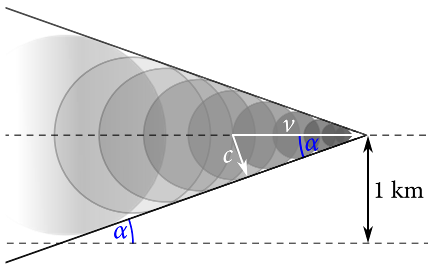


Figure 21.1: *Shock wave from Marc's airplane*

If Sabine with her aircraft were stationary, the sound envelope from Marc's aircraft would reach her only when Marcel was at a distance of

$$1\ \text{km}\cdot\cot\alpha = 1\ \text{km}\cdot\sqrt{\frac{1}{\sin^2\alpha}-1} = 2\sqrt{2}\ \text{km} \tag{21.1}$$

from the point of their closest approach. Since Sabine is moving at a speed v in the opposite direction, both would only manage to reach half the distance of $\sqrt{2}$ km. Marc covers this distance in time

$$\frac{\sqrt{2}\,\mathrm{km}}{1\,\mathrm{km/s}} = \sqrt{2}\,\mathrm{s}. \tag{21.2}$$

22 If the empty balloon with newlyweds weighs M and the hydrogen in it weighs m_H, the total gravitational force acting on the inflated balloon will be $(M + m_\mathrm{H})g$. For the balloon to float, an upward buoyant force $V\rho_\mathrm{a}g$ must act on it, where V is the volume of the balloon and ρ_a is the density of air.^[We silently assume the the volume of the newlyweds is negligible compared to this volume. The equation for the equality of forces is

$$(M + m_\mathrm{H})g = V\rho_\mathrm{a}g, \tag{22.1}$$

from which, substituting $V = \frac{m_\mathrm{H}}{\rho_\mathrm{H}}$, where ρ_H is the density of hydrogen, we get

$$M + m_\mathrm{H} = \frac{m_\mathrm{H}}{\rho_\mathrm{H}}\rho_\mathrm{a}, \tag{22.2}$$

and from there we can express the mass of hydrogen as

$$m_\mathrm{H} = \frac{M}{\frac{\rho_\mathrm{a}}{\rho_\mathrm{H}} - 1}. \tag{22.3}$$

Now we can find the density of hydrogen under normal conditions $\rho_\mathrm{H} = 0.09\,\mathrm{kg/m^3}$ or utilize the knowledge that under standard conditions, a mole of any gas occupies 22.4 l, meaning it has a molar volume of $V_\mathrm{m} = 22.4\,\mathrm{l/mol}$. Therefore, the density of hydrogen is

$$\rho_\mathrm{H} = \frac{m_\mathrm{H}}{V} = \frac{m_\mathrm{H}}{nV_\mathrm{m}} = \frac{M_\mathrm{H}}{V_\mathrm{m}}, \tag{22.4}$$

where $M_\mathrm{H} = 2\,\mathrm{g/mol}$ is the molar mass of molecular hydrogen. Equation 22.3 then transforms into

$$m_\mathrm{H} = \frac{M}{\frac{\rho_\mathrm{a}V_\mathrm{m}}{M_\mathrm{H}} - 1} \doteq 73.7\,\mathrm{kg}, \tag{22.5}$$

and therefore the amount of substance of hydrogen is

$$n_\mathrm{H} = \frac{M}{\rho_\mathrm{a}V_\mathrm{m} - M_\mathrm{H}} \doteq 36.9\,\mathrm{kmol}. \tag{22.6}$$

This hydrogen was produced electrolytically. The newlyweds are burning it, which is a process opposite to electrolysis. Therefore, if burning a mole of hydrogen releases energy H, it also takes energy H to produce it, at least in an ideal world. However, the newlyweds do not live in such a world, and their electrolysis apparatus has an efficiency η, meaning that to obtain a mole of hydrogen, $\frac{H}{\eta}$ energy is needed, and for n_H this is the energy

$$E = \frac{H}{\eta}n_\mathrm{H} = \frac{H}{\eta}\frac{M}{\rho_\mathrm{a}V_\mathrm{m} - M_\mathrm{H}}, \tag{22.7}$$

which for the values given is approximately 21 GJ or 5855 kWh. At the newlyweds' electricity price, this costs about 1171 €.

23 We see that between each pair of ranks in the ladder, we have a perfect conductor. Therefore, we can divide the ladder into a series connection of 99 parallel circuits, the total resistance of which we should already be able to calculate. The resistance of the n-th section of the ladder is calculated with the formula for resistance of parallel branches,

$$R_n = \frac{1}{\sqrt{n}}\,\Omega \parallel \frac{1}{\sqrt{n+1}}\,\Omega = \frac{1}{\frac{1}{\frac{1}{\sqrt{n}}\,\Omega} + \frac{1}{\frac{1}{\sqrt{n+1}}\,\Omega}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}\,\Omega. \tag{23.1}$$

Now we know we can reduce the connection to 99 resistors connected in series, and we need to sum their resistances. It does not look simple… fortunately, about the only sensible thing we can do is to expand each value with a suitable one to eliminate the square roots in the denominator:

$$R = \sum_{n=1}^{99} R_n = \sum_{n=1}^{99}\left(\frac{1}{\sqrt{n+1}+\sqrt{n}} \cdot \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}\right)\Omega = \sum_{n=1}^{99}\left(\sqrt{n+1}-\sqrt{n}\right)\Omega. \tag{23.2}$$

In this, we identify the so-called *telescopic series*: except for the last positive and the first negative term, everything else cancels each other out, and we get the result

$$\sum_{n=1}^{99}\left(\sqrt{n+1}-\sqrt{n}\right)\Omega = \left(\sqrt{100} - \cancel{\sqrt{99}} + \cancel{\sqrt{99}} - \sqrt{98} + \ldots - \cancel{\sqrt{2}} + \cancel{\sqrt{2}} - \sqrt{1}\right)\Omega = 10\,\Omega - 1\,\Omega = 9\,\Omega. \tag{23.3}$$

24 The sea level will behave over time as a sine, so let's draw one. The sine function has sa period od 2π; however, in our case, it will be scaled to 12 h. The question is, for what portion of the arguments of the function (horizontal axis) is the value greater than $\frac{2}{3}A$. This will not depend on whether the function period is 2π or 12 h – so for simplicity, let's draw one with the period of 2π.

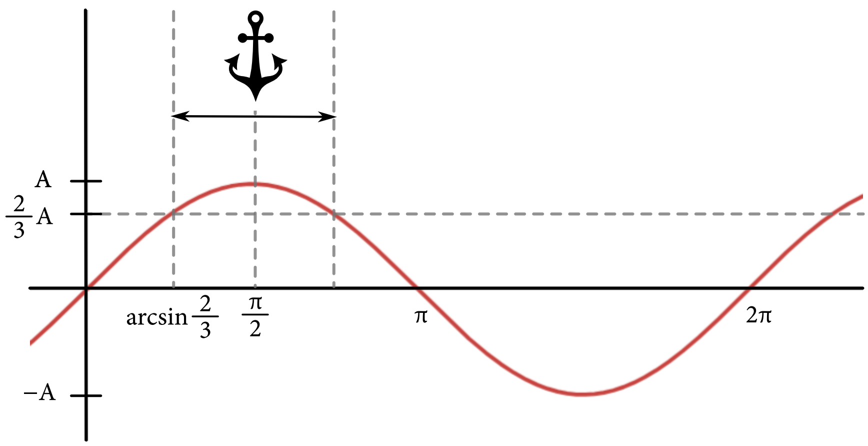


Figure 24.1: *The behavior of the sea level with the time suitable for anchoring marked*

From the image, it is clear that we will be interested in when the function reaches the value $\frac{2}{3}A$. We can calculate it exactly as $\arcsin\left(\frac{2}{3}\right)$.

What portion of one period of length 2π is between $\arcsin\left(\frac{2}{3}\right)$ and $\frac{\pi}{2}$? It is

$$\frac{\frac{\pi}{2} - \arcsin\left(\frac{2}{3}\right)}{2\pi} \tag{24.1}$$

and there are two such pieces, so the total probability is

$$2 \cdot \frac{\frac{\pi}{2} - \arcsin\left(\frac{2}{3}\right)}{2\pi} \doteq 0.267\,72 \doteq 26.8\,\%. \tag{24.2}$$

25 Let's stand in the centre of the Earth and mark the angle between points α. The distance measured by Marcel is $R\alpha$ and the distance measured by Max can be easily calculated as the base of an isosceles triangle, specifically

$$2R\sin\frac{\alpha}{2}. \tag{25.1}$$

These two distances should differ by $\Delta\ell = 1\ \mathrm{m}$. From this, we get the equation

$$R\alpha = 2R\sin\frac{\alpha}{2} + \Delta\ell. \tag{25.2}$$

We would like to solve this equation for α. This is not at all simple! In fact, it is analytically impossible! We will therefore have to settle for an approximate solution, for example through numerical binary search or using Taylor expansion. For the sine function, it holds that

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots. \tag{25.3}$$

Since $\Delta\ell$ is small, we should expect α to be small as well. We will only take the first non-trivial term and get

$$R\alpha = R\alpha - R\frac{\alpha^3}{24} + \Delta\ell, \tag{25.4}$$

from which we manipulate to express

$$R\alpha = R\sqrt[3]{\frac{24\,\Delta\ell}{R}} = \sqrt[3]{24R^2\,\Delta\ell}. \tag{25.5}$$

This is exactly the sought distance of Moritz. Substituting gives 99.2 km.

26 Only two forces are acting on Adam in this problem: gravity, with magnitude mg and tensile force from the chains of the swing, whose magnitude can change, but we know that it is always centripetal – in the direction towards the hinge. The g-force which Adam feels is always equal to the sum of all contact forces, which is in this case means only the tensile strength from the chains. All we need to do is to determine its magnitude at the lowest point.

In the left- and top-most point of his trajectory Adam is not moving at all. Since he does not stay there, he must be subject to some force; and since he is bound to the arc, the direction of this force must be tangential and it must point back towards the lowest point. At the same time, the tensile force must be $m \cdot 0.5\,g$, and the gravity is mg as always.

If we sketch these acting forces for a general angle of maximal deflection α, we see that the magnitude of the centripetal force is $mg\cos\alpha$. This means that in Adam's case α must be 60°, as $\cos 60° = 0.5$.

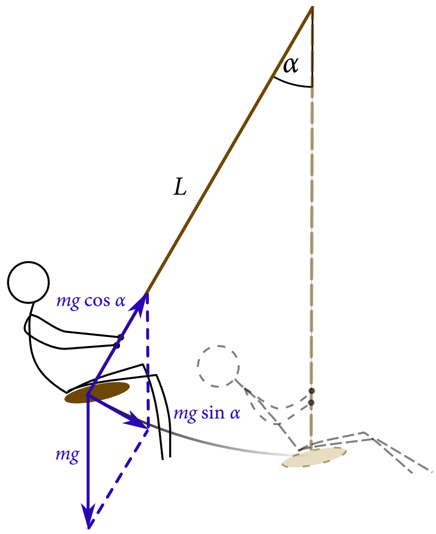


Figure 26.1: *Forces acting on Adam at maximal deflection*

Now we need to calculate Adam's speed at the lowest point. This comes from transforming the difference in potential energies between the highest and lowest points of his trajectory to kinetic energy, which is

$$\Delta U = mg(L - L\cos 60°) = mg\frac{L}{2}, \tag{26.1}$$

and so

$$\frac{mv^2}{2} = mg\frac{L}{2} \quad \Rightarrow \quad v = \sqrt{gL}. \tag{26.2}$$

Finally we find the magnitude of the tensile force from the chains and the corresponding g-force. Its origins are, again, twofold: centripetal force, which ensures that Adam will remain bound to the arc, and the reaction of the swing to Adam's gravity. Their sum is

$$F = ma = \frac{mv^2}{L} + mg = mg + mg = 2mg, \tag{26.3}$$

and the magnitude of apparent acceleration that Adam feels will thus be

$$a = \frac{2mg}{m} = 2\,g. \tag{26.4}$$

27 Faraday's law of electromagnetic induction states that the induced voltage is equal to the negative rate of change of the magnetic flux through a loop. Magnetic flux is calculated as $\Phi = BS\cos\theta$, where B is the magnitude of the magnetic field passing through area S, entering at an angle θ.

In our case, S is the inner area of the necklace. It is evidently on Kate's neck, and thus it has the shape of her neck, which does not change. The magnitude of magnetic induction B also does not change, according to the problem statement. And since Kate rotates only around the axis of rotation of the merry-go-round, the angle θ does not change either.

All of this means that the magnetic flux through Kate's necklace does not change over time, and therefore no voltage is induced in it, and consequently also no current.

28 Let us denote the buoy's base area S_b, its density ρ_b and its volume V_b; the opening's area S_o and the rope's length ℓ. Let the water level reach height h in the limiting situation. Rhe submerged part of the buoy is $v = h - \ell$ tall. The forces acting on the buoy must be balanced, therefore

$$F_G + F = F_{\mathrm{vz}}, \tag{28.1}$$

where F_G is gravity, F_{vz} is buoyancy and F is a force by which the rope pulls on the buoy. In the limiting situation, the force due to the rope is equal to the force due to pressure by which the plug is pushed to the opening,[4], i.e. $F = p_h S_o$, where p_h is hydrostatic pressure near the bottom. After substituting all forces in, we obtain

$$S_b H \rho_b g + h\rho g S_o = S_b(h-\ell)\rho g, \tag{28.2}$$

where ρ is water density.

The least water volume which can be poured in, apparently corresponds to the situation in which the buoy is tied to the bottom as tight as possible, thus $\ell \to 0$. Then we obtain

$$h = \frac{S_b}{S_b - S_o}\frac{\rho_b}{\rho}H \tag{28.3}$$

and the corresponding volume

$$V_{\min} = (S - S_b)h = \frac{S - S_b}{S_b - S_o}\frac{\rho_b}{\rho}V_b. \tag{28.4}$$

The largest possible water volume which can be poured in corresponds to the situation in which the plug is released just in the moment when the buoy is completely submerged, thus when $h = H + \ell$. It happens when the rope's length is

$$\ell = \left(\frac{S_b}{S_o}\frac{\rho - \rho_b}{\rho} - 1\right)H, \tag{28.5}$$

which corresponds to volume

$$V_{\max} = S\ell + (S - S_b)H = \left[\frac{S}{S_o}\left(1 - \frac{\rho_b}{\rho}\right) - 1\right]V_b. \tag{28.6}$$

According to the problem statement, $S_o = \frac{S}{4}$ and $S_b = 0.99S$. For the provided numeric values, it yields minimum volume $V_{\min} = \frac{1}{148}$ l and maximum volume $V_{\max} = 1$ l.

[4] If it were any larger, it would pull the plug out.

29 The problem statement tells us that during every third passage of the slower pendulum with period T_1 at the extreme right position, it meets the faster one with period T_2 (where $T_1 > T_2$). Between these two moments, the slower pendulum executes three periods and the faster some number of n periods, thus

$$3T_1 = nT_2. \tag{29.1}$$

The second important piece of information is that between these moments, the faster pendulum executes five periods and the slower one some number of m periods, so

$$mT_1 = 5T_2. \tag{29.2}$$

It does not matter at all whether the pendula meet on the left or right. They both perform a simple harmonic motion, and if they meet like this on the left and right, they will meet on both sides after completing five and three periods, respectively. Therefore, it is also unimportant whether Dan initially displaces the pendula to the left or right.

If we divide equations 29.1 and 29.2, we get

$$\frac{3}{m} = \frac{n}{5} \quad \Rightarrow \quad mn = 15. \tag{29.3}$$

One solution to this equation is $m = 15$, $n = 1$, but this solution does not make sense – from the first paragraph we know that if the slower pendulum moves for three periods and the faster n periods, then n must be greater than 3. The second solution to equation 29.3 is $m = 3$, $n = 5$, which meets our requirements. This means that the pendula have periods with lengths in ratio of 3 : 5.

The period of a pendulum with string length ℓ is

$$T = 2\pi\sqrt{\frac{\ell}{g}}. \tag{29.4}$$

If we write this for our pendula with lengths ℓ_1 and ℓ_2 and know the ratio of the periods, we have

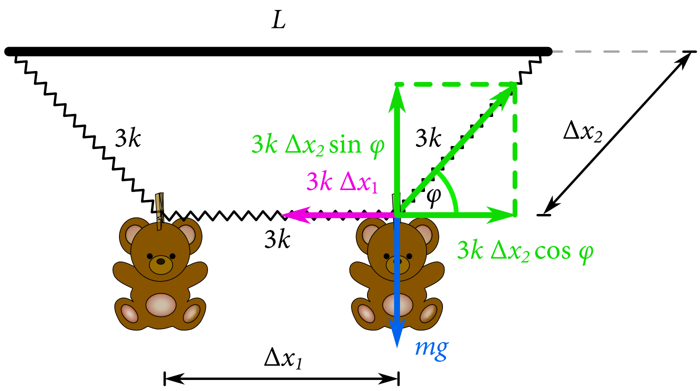
$$\frac{T_1}{T_2} = \sqrt{\frac{\ell_1}{\ell_2}}. \tag{29.5}$$

From the problem statement we know $\ell_1 = 10$ cm, and from equation 29.5 we express

$$\ell_2 = \ell_1\left(\frac{T_2}{T_1}\right)^2, \tag{29.6}$$

which for the ratio $T_2 : T_1 = 3 : 5$ gives us the result $\ell_2 = 3.6$ cm.

30 First, let's realize that if we have a spring with stiffness k and divide it into three equal pieces, each piece will have stiffness $3k$: if we stretch the original spring with a force F, each of its thirds will stretch by one third of the total stretch. However, the same force F acts on each third, which gives us triple stiffness.



At the point where the right plushies is hung, three forces act together. The first, gravity, has a magnitude of mg and points directly downward. The second, from the part of the spring between the plushiesh, points directly to the left and has a magnitude of $3k\,\Delta x_1$. The third generally forms an angle φ with the horizontal direction and has a magnitude of $3k\,\Delta x_2$. We can resolve the third force into a horizontal component $3k\,\Delta x_2 \cos\varphi$ and a vertical component $3k\,\Delta x_2 \sin\varphi$, where we know that the vertical component must equal mg and the horizontal one $3k\,\Delta x_1$. Notice that the distance the plushies drop is exactly $\Delta x_2 \sin\varphi$. We can express this value from the equality of vertical forces as

$$\Delta x_2 \sin\varphi = \frac{mg}{3k}. \tag{30.1}$$

31 As we have already shown in the sample solution of the task 16, the height of the triangle is $\frac{a\sqrt{3}}{2}$, the radius of the circle is $\frac{a}{2\sqrt{3}}$, and its centre is identical with the centroid of the triangle.

First, we will determine that if we connected a voltage source to the points between which we measure the resistance, no current will flow through the height of the triangle (the base rod). If we connected the voltage source in reverse, the current should flow in the opposite direction. However, we find that the configuration looks exactly the same as before, so the currents must flow the same way. This means that $I = -I$, which is only valid for $I = 0$.

By similar reasoning, we can determine that no current will flow between the circle and the triangle at their bottom point of contact. Since no current will flow through this node, we can disconnect it and simplify the scheme even further. After redrawing, we obtain the scheme shown in figure [@-fig:deathly-hallow-2:sol].

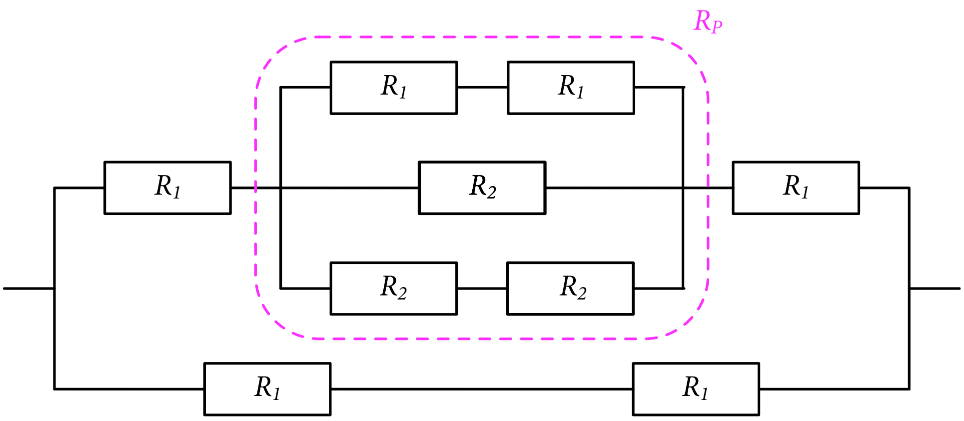


Figure 31.1: *Redrawing of the resistance network in Sarah's pendant*

We used the notation $R_1 = \frac{\lambda a}{2}$ for the resistance of half the side of the triangle and $R_2 = \frac{a\pi\lambda}{3\sqrt{3}}$ for the resistance of one-third of the circle. This is just a series-parallel connection of resistors, whose equivalent resistance we can calculate according to the known rules:

- the resistance of series-connected resistors is the sum of their resistances;
- the reciprocal value of the resistance of parallel-connected resistors is the sum of the reciprocals of their resistances.

If we denote the resistance of the trio of parallel branches as R_P, the following must be true:

$$\frac{1}{R_P} = \frac{1}{2R_1} + \frac{1}{R_2} + \frac{1}{2R_2} = \frac{3R_1 + R_2}{2R_1R_2} \quad \Rightarrow \quad R_P = \frac{2R_1R_2}{3R_1 + R_2}. \tag{31.1}$$

Subsequently, for the total resistance of the pendant we know that

$$\frac{1}{R} = \frac{1}{2R_1} + \frac{1}{2R_1 + \frac{2R_1R_2}{3R_1+R_2}} = \frac{6R_1 + 3R_2}{2R_1(3R_1 + 2R_2)} \quad \Rightarrow \quad R = \frac{2}{3}\frac{3R_1 + 2R_2}{2R_1 + R_2}R_1 = \frac{\lambda a}{6}\frac{9\sqrt{3} + 4\pi}{3\sqrt{3} + \pi} \doteq 0.563\lambda a. \tag{31.2}$$

32 When the gravitational force from the Sun disappears, the Earth will continue moving in a straight line at constant speed, equal to its original orbital speed. The trajectory will be tangent to the original orbit. If the gravity is turned off for long enough, it will certainly move too far to be captured by the Sun again. Between these extremes it will remain in a newer, more eccentric orbit.

The two different fates are separated by a critical situation in which the total mechanical energy after turning the gravity on again is zero. Since without gravity there is no potential energy here, the kinetic energy will remain constant too. Therefore we only need to find the time needed to get to a point where the kinetic and potential energy (after turning the gravity on again) sum to zero:

$$T + U(r) = 0. \tag{32.1}$$

The potential energy in a central gravitational field is

$$U(r) = -\frac{GM_\odot m_♁}{r}, \tag{32.2}$$

while the kinetic energy $T = \frac{m_♁ v^2}{2}$ remains constant. For a known radius of the orbit d we can express v from the identity of gravitational and centripetal force,

$$\frac{v^2}{d} = \frac{GM_\odot}{d^2} \quad \Rightarrow \quad v^2 = \frac{GM_\odot}{d}. \tag{32.3}$$

After plugging equations 32.2, 32.3 into equation 32.1 and dividing by the mass of the Earth $m_♁$ we get

$$\begin{aligned} \frac{GM_\odot}{r} &= \frac{v^2}{2}, \\ \frac{GM_\odot}{r} &= \frac{GM_\odot}{2d} \quad \Rightarrow \quad r = 2d. \end{aligned} \tag{32.4}$$

The Earth thus has to get at least twice as far as it was orbiting before the curse was cast. However as it is not moving radially, but tangentially at first, to find the length of its trajectory L we will use the Pythagorean theorem $L = \sqrt{(2d)^2 - d^2} = \sqrt{3}d$. Now we could express the orbital speed explicitly, however, it is easier to remember that one orbit takes exactly one year. Therefore

$$v = \frac{2\pi d}{1\,\mathrm{a}} \tag{32.5}$$

and the Dark Lord™ wil need to keep the gravity off for

$$t = \frac{L}{v} = \frac{\sqrt{3}d}{2\pi d} \cdot 1\,\mathrm{a} = \frac{\sqrt{3}}{2\pi}\,\mathrm{a} \doteq 101\,\mathrm{d}, \tag{32.6}$$

or a bit more than three months.

33 Absolutely black double hotplate heated to an astronomical temperature T_0 cools down by radiation. At the equilibrium temperature, it radiates with the power supplied from the electrical grid. That is

$$P = S\sigma T_0^4, \tag{33.1}$$

where S is its area and σ is the Stefan-Boltzmann constant.

When cooling it with a liquid, some portion of this power P' is used for heating and evaporation of the liquid, which we can specifically express as

$$P = S\sigma T_1^4 + P', \tag{33.2}$$

where T_1 is the temperature of the cooled double hotplate. The liquid receives power

$$P' = S\sigma(T_0^4 - T_1^4). \tag{33.3}$$

Over time t, the liquid absorbs heat $Q = P't$, which is utilized for heating and evaporating the liquid of mass m, mass heat capacity c, and specific latent heat of vaporization l by temperature ΔT, which we can specifically write as $Q = mc\,\Delta T + ml$. In our case, $\Delta T = 80$ °C, because the liquid is heated from 20 °C to 100 °C, and then it evaporates and leaves the surface. Thus we have

$$P't = mc\,\Delta T + ml. \tag{33.4}$$

We will write m as $V\rho$ and substitute for P' from equation 33.3, to express the volumetric flow rate

$$\frac{V}{t} = \frac{S\sigma(T_0^4 - T_1^4)}{\rho(c\,\Delta T + l)}. \tag{33.5}$$

For the values from the problem, this is 0.094 ml/s.

34 To answer the question, it is necessary to find the centre of mass of the paraboloid. Choose a coordinate system as indicated in the picture. Surface of the paraboloid can be described by function $y = k\sqrt{x}$. As point $[H; R]$ belongs to the paraboloid's surface, $R = k\sqrt{H}$, from where $k = \frac{R}{\sqrt{H}}$, thus $y = \frac{R}{\sqrt{H}}\sqrt{x}$.

The paraboloid's centre of mass lies on the paraboloid's axis at height v. This height can be found if we divide the paraboloid into reasonably chosen pieces, and the paraboloid's centre of mass will be a weighted average of centres of masses of individual pieces. In this particular case, a reasonable choice are very thin horizontal slices. Consequently, the x coordinate of the paraboloid's centre of mass is

$$x_T = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{\sum_i \sigma \pi y_i^2 \cdot x_i}{M} = \frac{\sum_i \sigma \pi \frac{R^2}{H} x_i \cdot x_i}{M}. \tag{34.1}$$

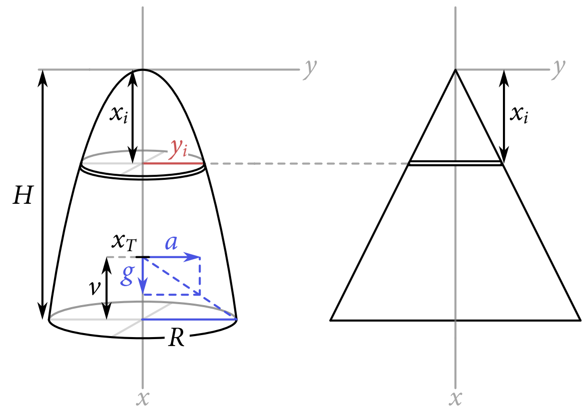


Figure 34.1: *A horizontal slice of a paraboloid and a triangle. In both cases, the mass of the slice increases linearly with the distance from the top, therefore both objects have their centre of mass at the same height.*

Notice that mass of individual pieces $m_i = \sigma \pi \frac{R^2}{H} x_i$ increases linearly with the coordinate. The same property poses also a triangle: if we cut it into slices – the length, and hence the mass of individual slices increases linearly with their distance from the apex.[5] However, we know that triangle's centre of mass lies at one third of its height, therefore the paraboloid must have the same property, hence

$$v = \frac{H}{3}. \tag{34.2}$$

Now, we only need to find a maximum admissible deceleration of the car. For this limit deceleration, the resulting acceleration (car's deceleration plus gravity) directs to the edge of the paraboloid's base. Based on the triangle similarity

$$\frac{a}{g} = \frac{R}{\frac{H}{3}} \quad \Rightarrow \quad a = \frac{3Rg}{H}. \tag{34.3}$$

35 Let's divide the Sun into a core and an outer layer, which is ejected during dying. Because during the ejection process, forces act only in the radial direction, it cannot change the angular momentum of the core. Therefore, we are only interested in the original angular momentum of the core, which will be conserved. We will need its mass and radius for its calculation.

From the density ratio in the problem statement, we know that

$$\rho_e = \frac{\rho_c}{63}, \tag{35.1}$$

[5] This is somewhat analogous to Cavalieri's principle

where ρ_c and ρ_e are the densities of the core and the envelope, respectively.

Furthermore, if we subtract the volume of the core from the volume of the Sun, we obtain the volume of the envelope

$$V_e = \frac{m_e}{\rho_e} = \frac{m_☼}{\rho_☼} - \frac{m_c}{\rho_c}. \tag{35.2}$$

If we use that $m_c = m_e = m_☼/2$ and substitute for ρ_e from equation 35.1, we obtain the density of the core

$$\rho_j = 32\rho_☼ = 32\frac{m_☼}{V_☼} = \frac{24}{\pi}\frac{m_☼}{r_☼^3}. \tag{35.3}$$

We calculate the radius of the spherical core from its mass and density as

$$r_c = \sqrt[3]{\frac{3}{4\pi}\frac{m_c}{\rho_c}} = \frac{r_☼}{4}. \tag{35.4}$$

Let us denote the moment of inertia of the core at the beginning as I_0 (when it has a radius of $r_☼/4$) and at the end as I. The conservation of angular momentum is expressed in the form

$$I_0\omega_0 = I\omega, \tag{35.5}$$

where $\omega_0 = 2\pi/T_0$ is the angular frequency of rotation, and $T_0 = 28$ d is the rotation period at the beginning, where $\omega = 2\pi/T$ is the final angular frequency and T is the required rotation period. By substituting and rearranging, we get

$$T = T_0\frac{I}{I_0}. \tag{35.6}$$

Thus, we only need to calculate the ratio I/I_0. The moment of inertia of a sphere with mass m and radius r is

$$I = \frac{2}{5}mr^2. \tag{35.7}$$

In our case, the core collapses into a smaller sphere, and thus only its radius changes. Therefore, in the ratio I/I_0, everything cancels except for the squares of the radii, so

$$T = T_0\frac{r_{WD}^2}{r_c^2} = 16T_0\frac{r_{WD}^2}{r_☼^2}. \tag{35.8}$$

where $r_{WD} = 5000$ km is the final radius of the white dwarf. After substituting the values, we get $T \doteq 1975$ s.

36 Let us denote the solar neutrino flux $\Phi = 10^{15}/(\mathrm{m}^2\cdot\mathrm{s})$. This is the number of neutrinos that pass through an area of 1 m^2 that is perpendicular to the line towards the Sun in 1 s. We can write

$$\Phi = \frac{\Delta N}{S\,\Delta t}, \tag{36.1}$$

where ΔN is the number of neutrinos, S is the area and Δt is the duration. Furthermore, let us denote the volume of the detector $V = 1000$ m^3 and the frequency of interactions between the neutrinos and the water

$R = 1\ \mathrm{s}^{-1}$. This is equal to

$$R = \frac{\Delta N_{\mathrm{int}}}{\Delta t}, \tag{36.2}$$

where ΔN_{int} is the number of neutrinos that react with water within a time interval of length Δt.

Since the neutrinos only interact very rarely, we can assume that the flux Φ is constant throughout the entire volume of the detector. Then we can imagine a huge amount of water through which the neutrinos are flying with a constant flux Φ, but instead of assuming that after a reaction a neutrino disappears (which would decrease the flux), we will assume that it continues its journey. Every neutrino thus flies through the detector undisturbed and interacts, on average, every time is passes distance ℓ.

Now imagine, all at once, the ΔN neutrinos that fly through a certain surface with area S in time Δt – this means, that on average every neutrino interacts exactly once in a volume of $S\ell$. Now we can express the quantity $\frac{R}{V}$, which represents the number of interactions per unit of volume and time, namely

$$\frac{R}{V} = \frac{\Delta N}{S\ell\,\Delta t} = \frac{\Phi}{\ell}. \tag{36.3}$$

From there we can immediately express

$$\ell = \frac{\Phi V}{R}, \tag{36.4}$$

and find the value $\ell = 10^{18}\ \mathrm{m} \doteq 106\ \mathrm{ly}$.

37 Let's start with Yusuf. Since he compressed the piston very quickly, we can assume that the heat exchange between the gas and the environment was negligible. Therefore, it was an adiabatic process. Let us denote the pressure of the air as p_{Y} and the volume of air after compression as $V_0 = \frac{V}{16}$. From the adiabatic law, we have

$$p_{\mathrm{Y}} V_0^{\kappa} = pV^{\kappa}, \tag{37.1}$$

where $p = 101\,325\ \mathrm{Pa}$ is the atmospheric pressure. From this we derive

$$p_{\mathrm{Y}} = p\frac{V^{\kappa}}{V_0^{\kappa}} = 16^{\kappa} p. \tag{37.2}$$

When fired, the gas expands adiabatically back to the original volume V. We can calculate the work done by the gas during the adiabatic process and set this work equal to the kinetic energy of the piston and the projectile,

$$\frac{1}{2}(m + M)v_{\mathrm{Y}}^2 = \frac{p_{\mathrm{Y}} V_0 - pV}{\kappa - 1}, \tag{37.3}$$

where v_{Y} is the speed of Yusuf's projectile. By substituting for p_{Y} from equation 37.2 and V_0, we obtain

$$v_{\mathrm{Y}} = \sqrt{\frac{2}{m + M}\frac{pV}{\kappa - 1}(16^{\kappa-1} - 1)} \approx 185.2\ \mathrm{m/s}. \tag{37.4}$$

Now let's look at Kim's shot. She also compressed the piston adiabatically, but then allowed it to cool to the ambient temperature at constant volume. However, since we are not interested in this process, but only its final state, we can treat it as an isothermal compression. Thus, the pressure p_{K} after compression can be calculated as

$$p_K V_0 = pV \quad \Rightarrow \quad p_K = p\frac{V}{V_0} = 16p. \tag{37.5}$$

Here, we must realize that in this case, during the shot, the process is again very quick and thus adiabatic. Atmospheric pressure is reached in the piston before the full volume V is achieved. The projectile is thus accelerated only until it reaches some partial volume V_1, where the pressures are equal (the pressure will thus equal the atmospheric pressure p). The piston will now start to decelerate while the projectile continues on its path. From the adiabatic law, we can derive

$$p_K V_0^\kappa = pV_1^\kappa, \tag{37.6}$$

and by substituting from equation 37.5, we get

$$V_1 = \left(\frac{p_K}{p}\right)^{\frac{1}{\kappa}} V_0 = 16^{\frac{1}{\kappa}-1} V. \tag{37.7}$$

Now we again calculate the work done by the gas as it adiabatically expands from volume V_0 to volume V_1 and set it equal to the kinetic energy of the piston and the projectile,

$$\frac{1}{2}(m+M)v_K^2 = \frac{p_K V_0 - pV_1}{\kappa - 1}. \tag{37.8}$$

By substituting $p_K V_0 = pV$ and V_1 from equation 37.7, we obtain

$$v_K = \sqrt{\frac{2}{m+M}\frac{pV}{\kappa-1}\left(1-16^{\frac{1}{\kappa}-1}\right)}, \tag{37.9}$$

which gives 96.1 m/s after substitution.

The quotient of v_Y to v_K is thus

$$\frac{v_Y}{v_K} = \frac{\sqrt{\frac{2}{m+M}\frac{pV}{\kappa-1}(16^{\kappa-1}-1)}}{\sqrt{\frac{2}{m+M}\frac{pV}{\kappa-1}\left(1-16^{\frac{1}{\kappa}-1}\right)}} = \sqrt{\frac{16^{\kappa-1}-1}{1-16^{\frac{1}{\kappa}-1}}},$$

which, after substituting $\kappa = 1.4$, results in $\frac{v_Y}{v_K} \doteq 1.93$.

38 Let's start by solving the geometry. If an equilateral triangle has a side length a, the inscribed circle has a radius $r = \frac{\sqrt{3}}{6}a$, and the height of this triangle is $v = \frac{\sqrt{3}}{2}a$.

The pendant consists of an equilateral triangle with a side length a, a circle with radius r, and a segment of length v. From the problem, we know that the axis of rotation passes through the entire segment, so its moment of inertia is $I_| = 0\,\mathrm{kg}\cdot\mathrm{m}^2$.

The moment of inertia of the circle is a greater challenge. We know that the moment of inertia of a body can be calculated by breaking the body into small pieces and summing the contribution of each piece to the total moment of inertia, thus $I = \sum_i m_i \rho_i^2$, where m_i is the mass of the i-th piece and ρ_i is the perpendicular distance of this piece from the axis of rotation. If the axis of rotation passes through the centre of the circle perpendicular to the plane of the circle, its moment of inertia would be

$$I_{\odot} = \sum_i m_i \left(x_i^2 + y_i^2\right) = mr^2. \tag{38.1}$$

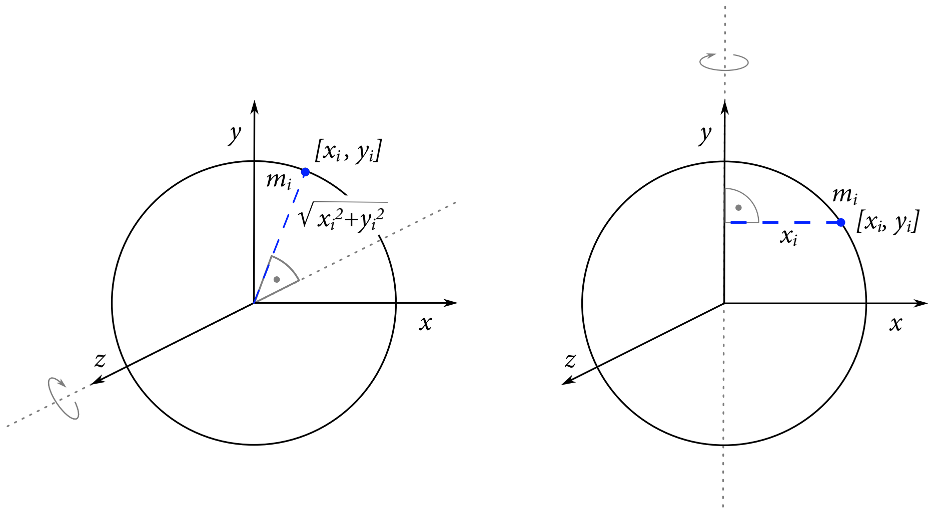


Figure 38.1: *The relationship between the moments of inertia for a circle rotating around two axes passing through its centre: axis perpendicular to its plane and an axis lying in its plane*

However, we have the axis of rotation located in the plane of the circle, thus

$$I_{\varnothing} = \sum_i m_i x_i^2. \tag{38.2}$$

When we realize that we can name the coordinates arbitrarily, obviously $2I_{\varnothing} = I_{\odot}$, which means

$$I_{\varnothing} = \frac{1}{2}mr^2. \tag{38.3}$$

The mass of the circle is $m = \lambda 2\pi r$, therefore

$$I_{\varnothing} = \lambda \pi r^3 = \frac{\sqrt{3}\pi}{72}\lambda a^3. \tag{38.4}$$

We still need the moment of inertia of the triangle. The moment of inertia of a rod perpendicular to the axis of rotation around its centre is $I_{-} = \frac{1}{12}m\ell^2 = \frac{1}{12}\lambda a^3$. When we realize that the moment of inertia of a point mass depends only on the perpendicular distance from the axis of rotation, in the case of a slanted rod we have to calculate the amount of mass of the rod at that distance. Here, this is very simple: mass is distributed homogeneously along their length, and so the moment of inertia of the empty triangle is the same as that of a single rod, but with three times more mass, so $I_{\triangle} = 3I_{-} = \frac{1}{4}\lambda a^3$.

Finally, we simply add the individual contributions to obtain the result

$$I = I_{|} + I_{\varnothing} + I_{\triangle} = \frac{\sqrt{3}\pi + 18}{72}\lambda a^3. \tag{38.5}$$

39 First of all, we must realize what we are going to calculate. Sound intensity level is a quantity which quantifies the energy per unit area and unit time carried by the sound wave. To be more precise, this is quantified by aound intensity I. Sound intensity level L is just a logarithm of ratio of sound intensity to some reference intensity I_0, and it is measured in bels, or in decibels:

$$L = \log\left(\frac{I}{I_0}\right)[\mathrm{B}] = 10\log\left(\frac{I}{I_0}\right)[\mathrm{dB}] \tag{39.1}$$

If a sound source emits acoustic power P, sound intensity at distance r from the source is $I(r) = \frac{P}{4\pi r^2}$, and the corresponding intensity level is

$$L(r) = 10\log\left(\frac{P}{4\pi I_0 r^2}\right) = 10\log(P) - 10\log(4\pi I_0) - 20\log(r). \tag{39.2}$$

Now we can start calculating. According to the problem statement, the intensity level equals Granny Julie's age every year. If Granny Julie celebrates L^{th} birthday this year, her age two years ago was

$$L - 2 = 10\log(P) - 10\log(4\pi I_0) - 20\log(r). \tag{39.3}$$

Last year, our neighbour came to help. If the neighbour can sing with acoustic power Π, granny's age must have been

$$L - 1 = 10\log(P + \Pi) - 10\log(4\pi I_0) - 20\log(r). \tag{39.4}$$

This year, we additionally move Julie's armchair by d towards the singing congratulants, and therefore her age is

$$L = 10\log(P + \Pi) - 10\log(4\pi I_0) - 20\log(r - d). \tag{39.5}$$

Subtracting equation 39.3 from equation 39.4 yields

$$1 = 10\log(P + \Pi) - 10\log(P) \quad \Rightarrow \quad P = \frac{\Pi}{\sqrt[10]{10} - 1} \tag{39.6}$$

and subtracting equation 39.4 from 39.5 yields

$$1 = 20\log(r) - 20\log(r - d) \quad \Rightarrow \quad r = \frac{\sqrt[20]{10}d}{\sqrt[20]{10} - 1}. \tag{39.7}$$

After substituting this into equation 39.3, we obtain

$$L = 2 + 10\log\left(\frac{\Pi}{\sqrt[10]{10} - 1}\right) - 10\log(4\pi I_0) - 20\log\left(\frac{\sqrt[20]{10}d}{\sqrt[20]{10} - 1}\right). \tag{39.8}$$

To determine Granny Julie's age, we must express the acoustic power of the invited neighbour in terms of sound intensity level which he can produce. If he can produce intensity level Λ at distance ρ, we can write

$$\Lambda = 10\log(\Pi) - 10\log(4\pi I_0) - 20\log(\rho). \tag{39.9}$$

After substituting into equation 39.8, we finally obtain

$$L = 2 + \Lambda - 10\log\left(\sqrt[10]{10} - 1\right) + 20\log\left(\frac{\sqrt[20]{10} - 1}{\sqrt[20]{10}} \cdot \frac{\rho}{d}\right), \tag{39.10}$$

which means that, according to the provided numeric values, Granny Julie celebrates her 99$^{\text{th}}$ birthday this year. Congratulations!

40 We will start by drawing all the forces acting on the cube and cylinder.

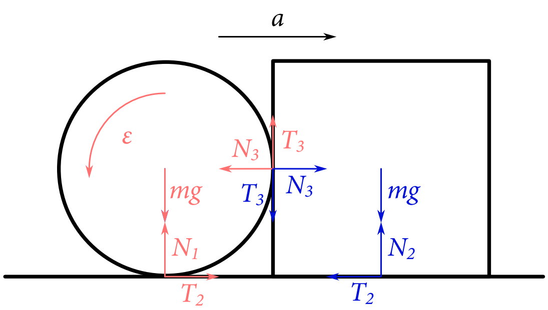


Figure 40.1: *Sketch of forces acting on the cylinder and cube*

Once we have the forces, we can write the motion equations. There will be five in total. Two for the motion of bodies in the vertical direction,

$$\begin{aligned} N_1 + f_3 N_3 &= mg, \\ N_2 - f_3 N_3 &= mg, \end{aligned} \tag{40.1}$$

two for the motion of bodies in the horizontal direction

$$\begin{aligned} ma - f_1 N_1 + N_3 &= 0, \\ ma + f_2 N_2 - N_3 &= 0, \end{aligned} \tag{40.2}$$

and one for the rotation of the cylinder

$$I\varepsilon - f_1 N_1 R - f_3 N_3 R = 0, \tag{40.3}$$

where a is the translational acceleration of the cylinder and cube. These must be equal, otherwise the cylinder would stop pushing the cube, and the fight would be over. $I = \frac{1}{2} m R^2$ is the moment of inertia of the cylinder, and ε is its angular acceleration.

By solving this gargantuan system of equations, we get

$$a = \frac{f(1-4f^2)}{2+f^2}g,$$
$$\varepsilon = \frac{4f+3f^2-4f^3}{2+f^2}\frac{2g}{R}. \tag{40.4}$$

When will the cube have the highest speed? The only force that can accelerate the whole system is the slip of the cylinder. When the cylinder stops slipping, the system will start to slow down due to the friction between the cube and the surface. The cylinder will stop slipping at time τ from the start, for which

$$a\tau - \Omega R + \varepsilon R\tau = 0, \tag{40.5}$$

which is the moment when the circumferential speed of a point on the surface of the cylinder matches its translational speed.

From there, we can express the maximum speed to which Danny is accelerated as

$$v = a\tau = a\frac{\Omega R}{\varepsilon R + a} = \Omega R\frac{1-4f^3}{9+6f-12f^2} \doteq 0.99\ \mathrm{m/s}, \tag{40.6}$$

which is about one meter per second.

Answers

1 2 Wh

2 6

3 1.3 °C

4 25 920 km/h^2

5 72 l

6 3700 km

7 Thomas, by 0.505 s

8 8

9 13 mm

10 −8.4 °C

11 $\frac{3}{4}$

12 $\arcsin \frac{2s}{gt^2}$

13 $\frac{3}{2}$ A

14 $\frac{400\pi}{3}$ m ≐ 419 m

15 2 s

16 $\frac{6 + \frac{2\sqrt{3}\pi}{3} + \frac{3\sqrt{3}}{4}}{6\sqrt{3} + 2\pi + 3} a \doteq 0.555a$

17 $L\sqrt{8\dfrac{M-m}{M+m}}$

18 60

19 $1.624 \cdot 10^{22}$ kg, *accept results in range* $1.62 \cdot 10^{22}$ kg – $1.66 \cdot 10^{22}$ kg.

20 180

21 $\sqrt{2}\ \mathrm{s} \doteq 1.41\ \mathrm{s}$

22 1171 €, *accept results in range* 1150 € – 1180 €.

23 $9\ \Omega$

24 $\dfrac{1}{2} - \dfrac{\arcsin\frac{2}{3}}{\pi} \doteq 26.8\ \%$

25 99.2 km, *accept results in range* 99 km – 99.3 km.

26 $2\,g$

27 0 A

28 $V_{\mathrm{min}} = \dfrac{1}{148}\ \mathrm{l}$, $V_{\mathrm{max}} = 1\ \mathrm{l}$

29 3.6 cm

30 $\dfrac{mg}{3k}$

31 $\dfrac{a\lambda}{6}\dfrac{4\pi + 9\sqrt{3}}{\pi + 3\sqrt{3}} \doteq 0.563 a\lambda$

32 $\dfrac{\sqrt{3}}{2\pi}\ \mathrm{a} \doteq 101\ \mathrm{d} \doteq 8.7 \cdot 10^{6}\ \mathrm{s}$

33 0.094 ml/s

34 $\frac{3Rg}{H}$

35 1975 s ≐ 33 min

36 10^{18} m ≐ 106 ly

37 1.93

38 $\frac{\sqrt{3}\pi + 18}{72}\lambda a^3$

39 99

40 0.99 m/s

Problems

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