Dear readers,

in your hands you are holding the booklet of the 28th volume of Náboj Physics. The booklet contains all the physics problems you could have encountered during the competition this year, as well as the solutions to them, from which you can learn a lot. If you have any difficulty understanding any of them, do not hesitate to contact us; we will gladly clarify everything.

This booklet would not exist without the enormous effort of many people who participated in organising Náboj Physics. Most of us are students of Faculty of Mathematics, Physics and Informatics of Comenius University in Bratislava, and some of us also actively participate in organising the Physics Correspondence Seminar (FKS).

Náboj Physics continues with its international tradition as well in the year 2025. For the international cooperation, we would like to thank to the local organisers: Katarína Nedeľková (Bratislava), Marián Kireš (Košice), Jakub Kliment (Prague), Lenka Plachtová (Ostrava), Ágnes Kis-Tóth (Budapest), Urszula Goławska (Gdańsk), Andrzej Karbowski (Toruń), Maciej Matyka (Wrocław), José Francisco Romero García (Madrid), Dmytro Rzhemovskyi (Vienna), Vasyl Rizak (Užhorod) and João Rico (Lisabon).The results of this international clash can be found on our web site.

In the name of the entire team of organisers, we believe that you enjoyed Náboj Physics in 2025, and we hope that we see each other at Náboj next year. Either as competitors or organisers.

Jaroslav Valovčan
Chief organiser

The results, the archive and other information can be found on the web site https://physics.naboj.org/.

Problems

1 Annie enjoys solving Sudoku puzzles from the Sunday newspapers. However, she doesn't find those puzzles challenging enough. So, she has come up with her own unique twist: she imagines that each side of a grid square measures 1 cm, and each square weighs exactly as many grams as the digit inside it. What will be the distance between the centre of mass of such a Sudoku grid and its geometrical centre after Annie solves the puzzle?

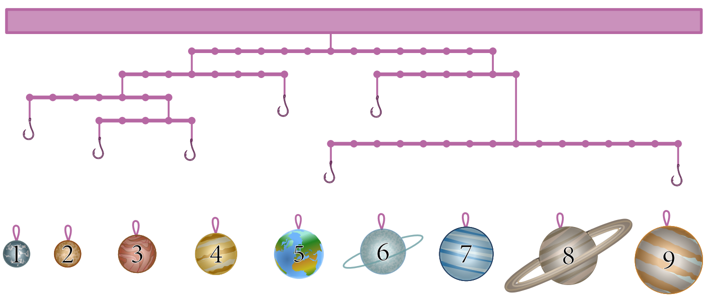
8	7	9		3		4		6
		6		7	9		1	
1			2	4	6	7		9
9		7	3			5	6	4
	2		6		4		9	7
	5			8		2		1
5	6	3	7		8	1	4	
7			4	1	2	6		
4	1		5	6			7	8

2 Jacob is sitting in a train compartment, enjoying his journey from Bratislava to Vienna. He faces away from the direction of travel, and in the mirror fixed to the opposite wall of the compartment, he sees the reflection of a signal post close to the tracks.

What is the apparent speed at which the image of the signal post approaches Jacob in the mirror, given that the train is moving forward at speed v?

3 On her zeroth birthday, little Helena got a cradle carousel. And not just a regular one, but with an astronomical theme: on several light arms of various lengths, different plush planets can be hung. But Martin, the proud father, remembered there used to be nine planets, and that is how many of them he ordered online. Their masses are 1 to 9 g.

Martin now needs to hang one planet on each hook, so that the whole carousel is balanced. Which planets will he **not** use?

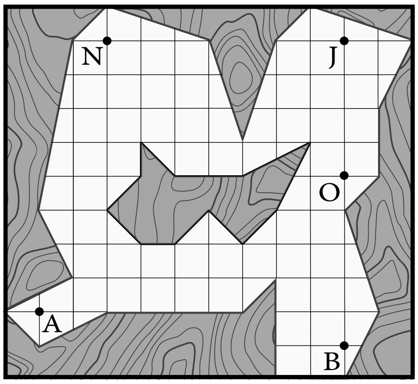


4 Joe went on a vacation to a nearby resort named Warm Hill. To his surprise, Warm Hill was not a hill, but a lake, and moreover the water was cold, hence he decided to take a bath at home instead. In the bathroom, he quickly identified two taps with warm and cold water.

Water flowing from the first one has temperature 68 °C. However, he would like the water to have temperature 34 °C and volumetric flow rate 600 ml/s. What volumetric flow rate should flow from the cold-water tap if the cold water has temperature 17 °C?

5 Nina (N), Adam (A), Barbara (B), Oliver (O), and Jane (J) are standing in a dark cave. Nina, the leader of the adventurous expedition, wants to find out where the others are, so she shouts her letter "N". Half a second after Adam hears her, he shouts "A", then Barbara does the same after hearing Adam's shout, and so on, until Nina hears Jane shout her letter. How long after Nina's shout will she hear the whole word "NABOJ" and know that nobody got lost?

The side of a square in the grid is 10 *metres, and the reaction time between the ears and mouths of all members of the expedition is the same,* 0.5 *s. Sound waves do not travel through solid rock.*



6 Patrick goes running on a track. Since he got a smartwatch, he always measures his average pace. During his last training, he noticed that between the completion of the third and the fourth lap, his average lap time improved by full two seconds from `01:20` to `01:18`.

How long did it take him to run the fourth lap?

7 Modern navigation systems come with all sorts of features. For example, they can even show the speed you need to maintain in order to arrive on time. Marcel is rushing to the airport and has set his arrival time to `22:00`. At `21:00`, the navigation tells him he should be driving at 120 km/h to make it. But by `21:30`, its estimate increased to 150 km/h.

When will Marcel actually arrive at the airport if he drives at a constant speed?

8 Sonya has bought a new glass lamp in the shape of a full cube. The cube has a refractive index of 1.5 and a light bulb in its centre. Excited to test her lamp, Sonya placed it on a stand in the exact centre of her room. What fraction of Sonya's room was illuminated by this lamp?

9 Simon is traveling by Shinkansen on his dream trip in Japan. The train he is on starts from the station with an acceleration of 2.6 km/h/s and has a maximum speed of 320 km/h. At what distance from the station will the train with Simon reach its maximum speed?

10 Vegan Lucy took an avocado with a toy inside with her on a trip. However, she was disappointed because the toy was again just an ordinary wooden ball with a radius 2 cm and a density 750 kg/m^3. Therefore she simply threw it into the lake. However, Mirko the water sprite immediately spotted it and pulled the ball to the bottom of the lake. How much work did he do on the ball if the lake is 30 m deep?

Neglect the work necessary to submerge the ball directly below the water surface.

11 Tram driver Jacob is approaching a level crossing with his tram at speed v. Suddenly, he notices that there is a car standing on the crossing. In a desperate attempt to prevent a collision, Jacob starts ringing the bell. He relies on the fact that when the driver in the car hears the bell, they can escape from the tracks within time t.

At what minimum distance before the crossing must Jacob ring the bell if the speed of sound is c?

12 Patrick used a huge paper to write down his notes in preparation for his final exams, so that he had everything in one place. The paper is so huge that it could cover entire floor of Patrick's room. He was relieved because if he kept writing and writing, he would have written so much, that the required paper would be sufficient to cover the entire Earth. What is the smallest A-format paper that can cover the entire Earth?

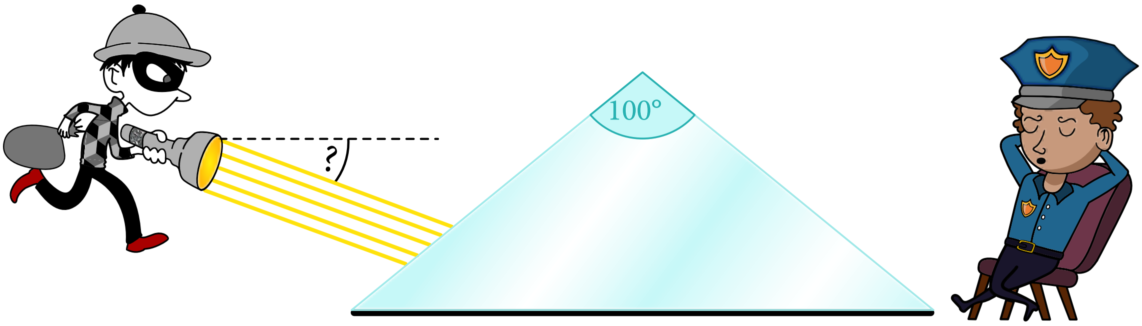
Paper `A0` has a surface area of 1 m^2. We can make `A1` paper by cutting the `A0` paper in half along the shorter edge, and so on. Patrick can freely cut the paper during his attempt to cover the Earth.

13 Sabine attended an air show. She felt rather cold, because wind was blowing from the left into her face at speed v the whole time. What fascinated her the most were the aerobatic jets. At the end of the show, one of the jets flew from right to left in front of the grandstand at speed u relative to the ground. Throughout its flight in front of the grandstand, it released colored smoke, which, relative to the air, quickly stopped moving.

How long was the smoke trail if the length of the grandstand was s?

14 Robert the robber wants to steal a triangular prism with refractive index 2.4 from the museum. The cross-section of the prism is an isosceles triangle with apex angle 100°, as shown in the picture. The robber will probably need to shine a parallel beam of light on it, but he does not want the museum guard on the opposite side of the prism to notice. At least by how much must he tilt the flashlight downward relative to the horizontal direction, so that no rays at all will pass through to the other side of the prism, no matter which part of the prism's face he shines on?

The bottom of the prism is perfectly black and absorbs all rays that fall on it.



15 Kai is setting up a gym at home. Right now, he is moving a new dumbbell weighing 20 kg up to his apartment in an elevator weighing 250 kg. Since he hasn't been going to the gym much yet, Kai weighs only 60 kg and his arms are so light that we can ignore their weight altogether. Despite the elevator rises at considerable constant speed 4 m/s, Kai does not want to lose any time, and he starts lifting the weights right away.

How much force is exerted on the rope holding the elevator when Kai moves the dumbbell downwards with acceleration $5\ \mathrm{m/s^2}$?

16 It's half past three in the morning and Sam is throwing eggs out the window, aiming at an unseen trombone player hiding somewhere in the shadows. When he throws an egg upwards, it lands in four seconds. If he throws it downwards with the same speed, two seconds are enough.

How long will the egg fall if Sam just drops it?

17 Lucy needed to keep her noisy nephew entertained, at least for a little while. When toys, building sets, and even threats with a wooden spoon didn't work, she pulled out a weightless rope and a weightless pulley from the cupboard and tied two weights to the ends of the rope – one with mass m and the other with mass $\frac{m}{2}$. They hung the rope over the pulley together and waited to see what would happen. The nephew watched with excitement for a moment as the weights started moving, but soon ran off yelling to cause more mischief elsewhere. Still, one question kept bothering Lucy:

"With what acceleration are the weights actually moving?"

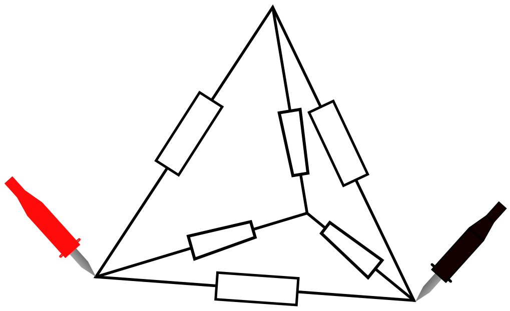
Help Lucy find an expression for the acceleration of the weights.

18 Submit the sum of the numbers next to correct statements.

1	A body is on a circular orbit. When its speed increases, its orbital period is shortened.
2	When the Earth gets closer to the Sun on its orbit, it is moving faster.
4	Winter occurs because the Earth's orbit is eliptical, and the Earth is at its farthest point from the Sun at that time.
8	Every satellite orbiting the Earth at a circular orbit is moving at the same speed.
16	Geostationary satellites can exist only above the equator.

32	In a two-body system, the lighter body has always an eliptic trajectory around the common centre of mass.

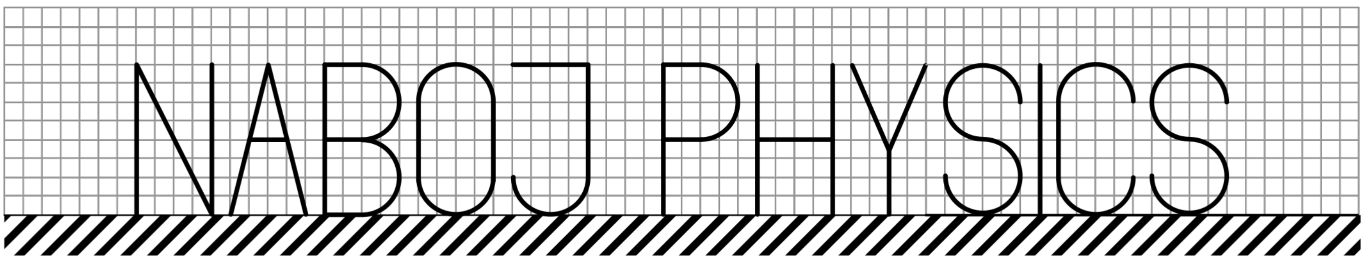
19 We have made a regular tetrahedron using six resistors with electrical resistance R. Subsequently, we connected an ohmmeter to two of its vertices. If we completely removed one branch with the resistor from the tetrahedron, what would be the total resistances that we could measure?



20 We need your help to sort the letters in our contest's name in order of their stability. By stability, we mean the amount of work required to tip a letter over.

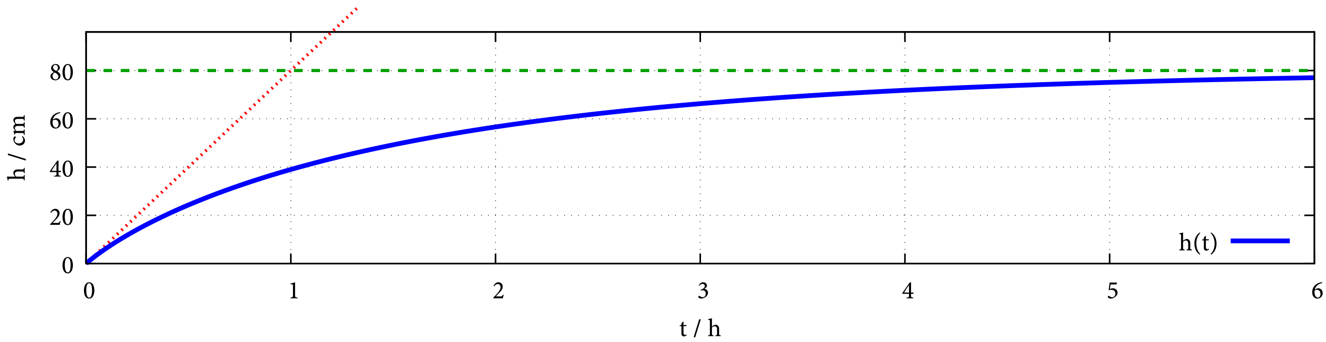
The letters are composed of homogeneous solid rods shaped as line segments or arcs of circles, with a consistent but small ***nonzero*** *thickness and their edges are sharp. The letters must stay in the plane of the paper.*

Submit a sequence of letters in ***descending*** *order. Letters that are not in a stable position should* ***not be included****.*



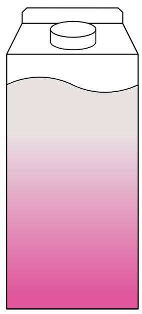
21 Adam the gardener has a huge cylindrical water tank for collecting rainwater. The area of its base is 1 m^2. After the long summer droughts, the tank was empty, but now it started raining, and the tank is filling up nicely.

In the tank, there is a water level sensor, and Matt can watch the height of the water level on his smartphone. At first, he was happy, since the water level seemed to be rising rapidly (see the dotted line, which is tangent to the blue line at $t = 0$). However, after a few hours, it became clear that the water tank will never be filled completely, but the rate of water level increase tapers off to zero at $h_{max} = 80$ cm (dashed line). Apparently, there is a hole in the bottom. What is the area of its cross section?



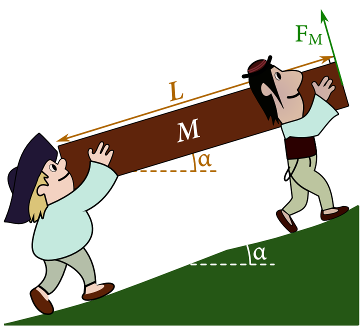
22 Mike bought a litre of strawberry yoghurt milk with 10 % of the volume being yoghurt (and 90 % water) and put it in the fridge. However, by morning, the milk had settled: at the top there was pure water, and the concentration of yoghurt increased linearly towards the bottom.

In the morning, Mike wanted to drink it from a glass with a volume of 400 ml, but forgot to shake the box. He only noticed this after he had already poured 200 ml. Then he closed the box again, shook it violently and filled the glass to the top. What is the concentration of yoghurt in Mike's glass of milk now?



23 Matthew and Jacob are carrying a log up an incline plane with slope α. The log is shaped like a long, thin cylinder with length L and mass M. Matthew carries the front end, and Jacob carries the rear end. However, Matthew is not pulling his weight: the force he is exerting on the log is not pointing up, but perpendicularly to the cylinder's axis. What is the ratio of the forces Matthew and Jacob exert on the log?

Matthew and Jacob are of the same height and they carry the log at the same height above the surface of the hill.



24 What is the fraction of the Earth's surface, where it is possible to see the Sun exactly in the north (at any altitude above the horizon)?

Assume that the Earth is a perfect sphere and that the solar rays are parallel. Ignore atmospheric refraction and similar effects.

25 Dan is preparing for the world championships in archery. The target is at a distance 70 m and is at the same height as Dan. How high above the centre of the target should Dan aim in order to hit the bullseye, if the arrow leaves Dan's bow with speed 50 m/s?

In theory, there are two ways to achieve this. In competitive archery, however, one always shoots directly, so submit the smaller of the two solutions.

26 When measuring things, Americans will use absolutely anything but SI units. John Doe created his own system of measurement, based on units *gulp*, *belch* and *wink* defined as follows:

- 1 gulp is the volume he can swallow at once, and he can down one pint in 20 gulps;
- 1 belch is the intensity of the sound of his belch, which corresponds to sound intensity level 100 dB;
- 1 wink is the time he needs to wink, and he can wink exactly twice in a second.

What is his own mass in his system of units if in US customary units it equals 200 pounds?

0 dB corresponds to a power flux of $1\ \mathrm{pW/m^2}$.

27 Justine needs to refill her 10 l nitrogen tank at work. As per usual, at half past four in the morning, the nitrogen man arrived. He brought five smaller 2 l nitrogen tanks pressurised to 5 MPa in his van.

The workflow is as follows: the nitrogen man connects two tanks with a valve. He then opens the valve to equalise pressures in the tanks. Once the nitrogen temperature is in equilibrium with the outside temperature, the tanks are disconnected from each other.

What is the maximal pressure the nitrogen man can achieve in the large tank?

28 History repeats itself. This year, Patrick discovered a strange black disc in his grandfather's cabinet again. Before he could serve a soy burger on it, or use it instead of a frisbee, his grandfather showed him that if he places it on the turntable platter, presses a few buttons and positions a needle in the spiral groove at the outer edge of the disc, ominous crackling and hissing sounds will start coming out from the speakers, along with some prehistoric music.

This time, Patrick wants to know how long it takes for the record to spin up from rest to its usual speed of 33⅓ revolutions per minute. Assume that he places the needle on the disc only after it has reached its standard speed.

The record is a solid disc with diameter 30 cm and mass 200 g. It lies on the platter with the same diameter and mass 2 kg, which rotates together with the record. The turntable is powered by an AC/DC adapter with output voltage 5 V and direct current 100 mA.

Neglect any energy losses.

29 Rizzimus Prime was playing with a charged skibidi cube. The cube was dripping one charge of $q = 1\,\mathrm{C}$ in each of its vertices. Rizzimus Prime spawned a ninth charge, which he dropped to chill in the centre of the cube, fr fr. Rizzimus Prime is lowkey sigma, so he knows exactly how big this charge needs to be to make sure no charge zooms off to Ohio or crashes into the centre of the cube, 'cause that would be straight-up cringe.

What does this charge have to be?

`#electrostatics #skibidi #naboj-physics #nocap`

30 The FKS space company announced a major success – a landing on the Moon. In reality, however, they didn't have enough funding, so they only filmed a video on the Earth. And they did it with nothing more than an ordinary phone, whose camera records at a frame rate of 60 Hz. To start with, they tried filming a projectile motion using a wrench.

At what frame rate should the video be played to make it look convincing? The acceleration due to gravity on the Moon is 16 % of that on the Earth.

31 Adam the gardener and grandma Justine, the famous herbalist, are neighbours and their houses share a common wall. Adam needs his cellar to be cool, so that his vegetables don't spoil and rot, while grandma Justine needs her cellar to be warm, so that her herbs dry properly.

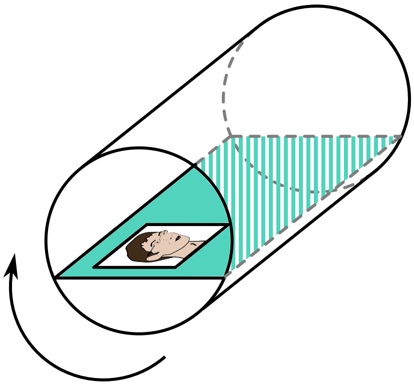
Therefore, they decided to invest in a heat pump. The heat pump is powered by electricity and draws heat from Adam's cellar to Grandma Justine's cellar. They two are very delighted with how their investment is paying off. Grandma Justine observes that the average efficiency of the heat pump is 1500 %.

What is the average efficiency observed by Adam?

32 Recently, Stanley read about the International Space Station. Therefore, he knows its mass is 400 t, and it orbits the Earth on a circular orbit at a height of 400 km above the surface. Despite being in heavenly heights, it is still affected by the atmospheric drag force, which causes it to descend by about 100 m closer to the surface each day.

With what force does the atmosphere slow down the International Space Station?

33 Adam once again spent a long time on the toilet. To pass the time more quickly, he began playing with a roll of toilet paper. He took his ISIC card out of his wallet and placed it inside the horizontally-held roll at the bottom. Then he rotated the roll. What was the maximum angle by which he could rotate the roll, so that the ISIC card would not slide down? The width of the ISIC card is $\sqrt{2}$ times greater than the radius of the roll, and the friction coefficient between the ISIC card and the roll is 0.3.



34 George often reminisces about his childhood – about the days when rainbows were still black and white, when there were no mobile phones, and he used to collect cards with pictures of horses. Among other things, each card showed the time required for that particular horse to accelerate from 0 km/h to 100 km/h on a flat surface.

Today, the rainbows are colourful, George also has a smartphone and collects cards of modern cars. The one he values the most shows Rimac Nevera, the fastest production car. Its electric motors can provide up to 1888 hp and accelerate the car to 100 km/h in 1.8 s. The limiting factor preventing it from accelerating even faster is not insufficient power, but the friction between the wheels and the road.

What is the shortest time it takes Rimac Nevera to accelerate from 0 to 100 km/h down an inclined plane if the motors can provide unlimited power and we can choose an arbitrarily steep hill?

35 On the western shore of a certain country in the tropics, there is a vertical rocky cliff. When the Sun sets on the shore, the top of the cliff is still illuminated: the Earth is round and the horizon is further away when we are looking from the top of the cliff than from the beach, and the Sun can be seen a little bit longer.

After the sunset, the edge of the shadow crawls up the cliff face at an uniformly accelerating rate. What is the magnitude of this acceleration?

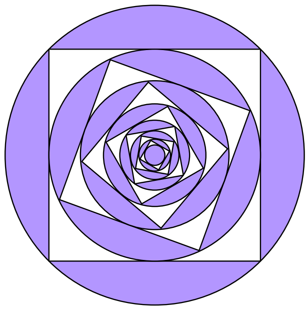
The cliff is located exactly on the equator. Assume it is the day of the vernal equinox.

36 Jacob visited the local nuclear power plant. The guide told many interesting facts about the operation of the plant to the visitors. For instance, Jacob learnt that natural uranium contains 0.72 % of ^{235}U and the rest is mostly ^{238}U, with half-lives of $7 \cdot 10^8$ and $4.46 \cdot 10^9$ years respectively. However, the reactor needs enriched uranium, which contains 3 % of ^{235}U.

Jacob is now curious: how much higher is the activity of enriched uranium compared to the same number of atoms of natural uranium?

37 Mom has given Sonya a harmonizing accessory – a necklace with an amulet that is shown in the figure. It is a circle of radius r, from which she has cut out an inscribed square. Inside this square, she has placed an inscribed circle. From that circle, she has cut out another inscribed square, and she has continued this process indefinitely. To make the amulet more powerful, she has rotated each inscribed square by 20° with respect to the previous one.

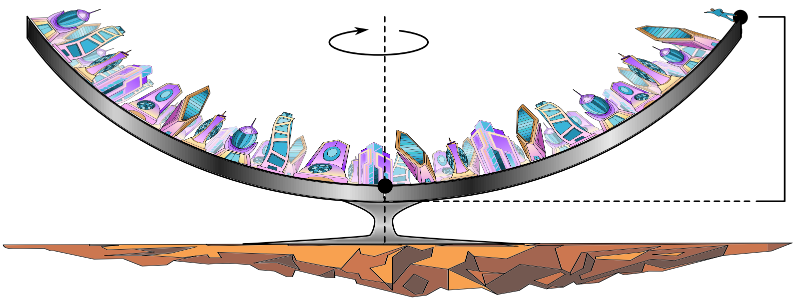
Sonya knows that the longer the amulet spins, the better it works. What is the moment of inertia of this object about the axis passing through the centre of the amulet and perpendicular to its plane? The mass of the amulet is m.



38 In the future, we are planning to colonise and terraform Mars. But the surface gravity on Mars is only 36 % of that on the Earth, and many people do not thrive under these conditions.

Therefore, we built Gyropolis, a massive city in the shape of a paraboloid of revolution with radius 1000 m, that provides artificial gravity by slowly rotating about its vertical axis. When an immigrant arrives on Mars, they get a room on the very edge, where the apparent gravity is equal to that on the Earth, and then slowly moves to the lower parts of the city, until they get used to lower gravity. Of course, at every point of the city, the direction of the perceived gravity is perpendicular to the surface.

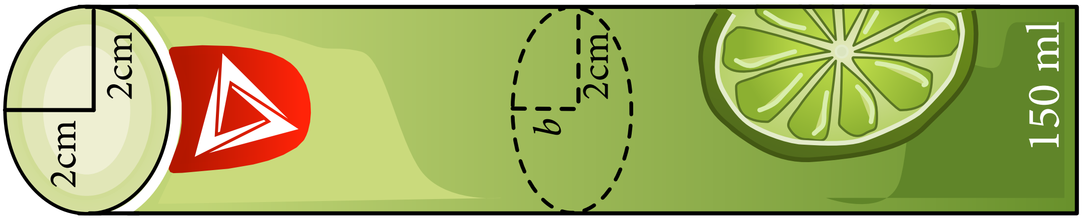
What is the difference in altitude between the centre and the edge of the city?



39 George needs a refreshment after a long hot summer day of cycling. In the nearest grocery store, he bought an ice lollipop. The lollipop is in a package shaped like a "deformed cone". The package has a volume of 150 ml and its horizontal cross section is elliptic at any height. The semi-major axis is constantly $a = 2$ cm, but the semi-minor axis is linearly changing with height – at the very top it is $b = a$ while at the very bottom it is $b = 0$.

George is slowly pushing the lollipop upwards and always immediately eats the part that protrudes out of the package. What work will he have done to push the lollipop upwards by the time he eats half of its volume? The density of the ice lollipop is 1000 kg/m^3.

The picture is rotated so it fits this paper, but even George holds the lollipop like any sane human being, with the opening at the top.



40 Tom loves fast cars and drag racing. During his last attempt to break the world record, his fans were uniformly spaced along the drag strip with length s. Tom put the pedal to the metal and ran down the strip with constant acceleration a. Each of his fans wrote down the instantaneous speed of Tom's car as it passed by them. After the race, the fans met in a pub, compared their notes and found the average of the measured values.

What is the ratio of this value to Tom's actual average speed during the race?

Solutions

1 To solve this problem, Annie doesn't need to complete it at all. Since, according to the rules, each number between 1 and 9 must appear exactly once in each column, all columns have the same weight (45 g) and the horizontal coordinate of the centre of gravity must therefore be exactly in the middle. The same argument applies to the rows and the vertical coordinate, so the centre of gravity is exactly in the middle, and the answer we are looking for is 0 cm.

2 For the purposes of solving the problem, we can imagine that the train (and therefore Jacob and the mirror) are stationary and the signal is moving towards it at a speed of v. The mirror transforms the scenery behind Jacob, so that objects at a distance d from the mirror will be seen by Jacob as being located d behind the mirror. Therefore, if the signal is moving towards him at a speed of v, its image in the mirror also moves towards Jacob at a speed of v.

3 Let's label the hooks from left to right with letters A to G. Note that the ratio of the distances of hooks E and G from the axis of rotation is $\frac{8}{7}$. The only two planets offered, whose masses are in the ratio 7 : 8, are precisely the 7- and 8-gram planets. We must therefore hang these planets on hooks E and G.

The total mass on these scales is therefore $7\ \text{g} + 8\ \text{g} = 15\ \text{g}$. The torque exerted on the arm above will therefore be $15\ \text{g} \cdot g \cdot 1\ \text{cm} = 15\ \text{g} \cdot \text{cm} \cdot g$[1]. In order to balance this torque with hook F at a distance of 5 cm from the axis of rotation, there must be a planet with mass 3 g.

Now let's note that the ratio of the distances of hooks B and C from the axis of rotation of their scales is 3 : 1. The only remaining planets whose masses are in this ratio are those with 2 g and 6 g. Together, they exert a torque on the arm above them $8\ \text{g} \cdot g \cdot 1\ \text{cm} = 8\ \text{g} \cdot \text{cm} \cdot g$, therefore hook A must accomodate the planet with mass 4 g. The planets on hooks A, B and C together weigh 12 g, so they exert a total torque of $12\ \text{g} \cdot g \cdot 3\ \text{cm} = 36\ \text{g} \cdot \text{cm} \cdot g$. The planet on hook D must balance this, so one with mass 9 g must be hung here.

From left to right, the masses of the planets must be 4, 2, 6, 9, 7, 3 and 8 g, and Martin will not use the planets with masses 1 g and 5 g. Which makes sense: Pluto is not a planet anymore, and they already have Earth at home anyway.

[1] For simplicity, we assume that the distance between each pair of points is 1 cm, but we could also choose any other nonzero constant.

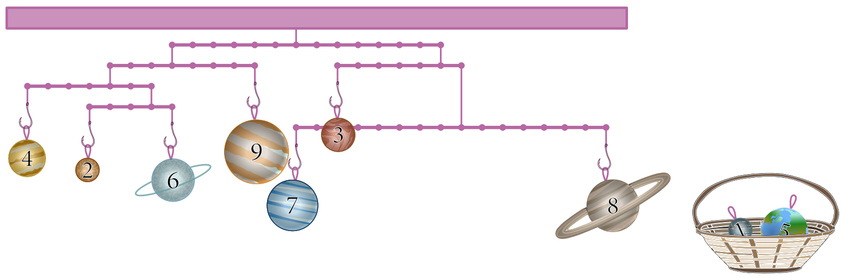


Figure 3.1: *Balanced scales*

4 The warm water transfers heat to the cold water until thermal equilibrium is reached. According to the problem statement, this equilibrium should occur at temperature 34 °C. We begin by writing the calorimetric equation

$$Q_{\text{absorbed}} = Q_{\text{released}}$$
$$m_{\text{cold}} \cdot c_{H_2O} \cdot (t - t_{\text{cold}}) = m_{\text{warm}} \cdot c_{H_2O} \cdot (t_{\text{warm}} - t). \tag{4.1}$$

Next, we can divide the entire equation by the specific heat capacity of water c_{H_2O}. Let x be the volume of cold water flowing into the bathtub per second. Since the total volumetric flow rate must be 600 ml/s, the volumetric flow rate of warm water will be 600 ml/s − x. We substitute these values into the equation and simplify,

$$x \cdot \rho_{H_2O} \cdot (34\,°\text{C} - 17\,°\text{C}) = (600\ \text{ml/s} - x) \cdot \rho_{H_2O} \cdot (68\,°\text{C} - 34\,°\text{C}),$$
$$x \cdot 17\,°\text{C} = (600\ \text{ml/s} - x) \cdot 34\,°\text{C}. \tag{4.2}$$

From this expression, we can solve for x,

$$x = \frac{600\ \text{ml/s} \cdot 34\,°\text{C}}{17\,°\text{C} + 34\,°\text{C}} = 400\ \text{ml/s}. \tag{4.3}$$

Therefore, the required volumetric flow rate of cold water is 400 ml/s.

5 Figure 5.1 shows the shortest path along which sound travels between every pair of consecutive friends. In some places, there is a stone wall in the direct path, which the sound must go around.

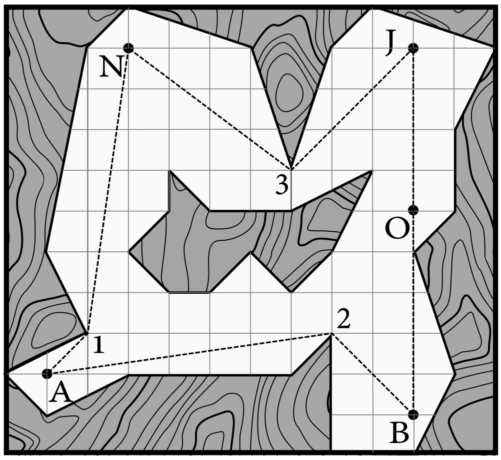


Figure 5.1: *Path of a sound wave in the cave*

We must therefore divide the first part of the path between points N and A into two shorter sections with an auxiliary point 1. Fortunately, on a square grid, all lengths either lie directly on the axes, or are hypotenuses of right triangles, so we only need to know Pythagoras' theorem.

For instance, the length of the first part is

$$d_{\mathrm{NA}} = d_{\mathrm{N1}} + d_{\mathrm{1A}} = \sqrt{(70\ \mathrm{m})^2 + (10\ \mathrm{m})^2} + \sqrt{(10\ \mathrm{m})^2 + (10\ \mathrm{m})^2} = \sqrt{5000}\ \mathrm{m} + \sqrt{200}\ \mathrm{m} = 50\sqrt{2}\ \mathrm{m} + 10\sqrt{2}\ \mathrm{m} = 60\sqrt{2}\ \mathrm{m}. \tag{5.1}$$

Similarly, we go through all sections one by one and obtain their lengths

$$d_{\mathrm{A2}} = 50\sqrt{2}\ \mathrm{m}, \quad d_{\mathrm{2B}} = 20\sqrt{2}\ \mathrm{m}, \quad d_{\mathrm{BO}} = 50\ \mathrm{m}, \quad d_{\mathrm{OJ}} = 40\ \mathrm{m}, \quad d_{\mathrm{J3}} = 30\sqrt{2}\ \mathrm{m}, \quad d_{\mathrm{3N}} = 50\ \mathrm{m}. \tag{5.2}$$

If we add up all the length, we get the total distance $\left(140 + 160\sqrt{2}\right)$ m. Sound travels at a speed of approximately $c_s = 343$ m/s, so to travel this distance, it will take

$$t_s = \frac{\left(140 + 160\sqrt{2}\right)\ \mathrm{m}}{343\ \mathrm{m/s}} \doteq 1.07\ \mathrm{s}. \tag{5.3}$$

However, we know that the reaction time of the expedition members is half a second for everyone. Since the shout passes through the four expedition members in this way, their total reaction time is $t_r = 4 \cdot 0.5\ \mathrm{s} = 2\ \mathrm{s}$, and Nina will therefore hear the entire word NABOJ after $t_s + t_r \approx 3.07$ s.

6 If Patrick's average lap time for the first three laps was 80 s, then the total time for those three laps is $3 \cdot 80\ \mathrm{s} = 240\ \mathrm{s}$. After the fourth lap, his average lap time is 78 s, thus the total time for all four laps is $4 \cdot 78\ \mathrm{s} = 312\ \mathrm{s}$.

Therefore, he ran the fourth lap in 312 s − 240 s = 72 s, or `01:12`.

7 For the first time, the navigation tells Marcel that he should go at a speed of 120 km/h, from which we can directly calculate Marcel's distance from the airport at `21:00`,

$$s = v \cdot t = 120\ \text{km/h} \cdot 1\ \text{h} = 120\ \text{km}. \tag{7.1}$$

In the same way, we can also calculate Marcel's distance from the airport at `21:30`,

$$s = 150\ \text{km/h} \cdot 0.5\ \text{h} = 75\ \text{km}. \tag{7.2}$$

From this, we know that in the first half hour of the drive, he travelled 120 km − 75 km = 45 km, thus the speed at which Marcel drives is $\frac{45\ \text{km}}{0.5\ \text{h}} = 90\ \text{km/h}$. Therefore the trip will take

$$\frac{120\ \text{km}}{90\ \text{km/h}} = 80\ \text{min}, \tag{7.3}$$

and he will arrive at the airport at `22:20`.

8 If the light bulb were shining without being inside the cube, it would illuminate the entire room. If we place a cube with a refractive index greater than 1 (possibly even other than 1.5) around the light bulb, according to Snell's law, the light will refract from the perpendicular line to the interface between the cube wall and the air. Therefore, if we look at a specific wall of the cube, the rays will be more scattered when they hit it than without the glass, meaning that they will illuminate more and certainly not less. Thus, 100 % of the room will still be illuminated.

9 The most important part of the problem is to notice the units of both quantities. The speed $v_{\max} = 320\ \text{km/h}$ is not compatible with the given units of acceleration 2.6 km/h/s. However, we can convert the latter to usable units $\text{km}/(\text{h} \cdot \text{h}) \equiv \text{km/h}^2$. If we know that the train accelerates by 2.6 km/h/s every second, in one hour (which consists of 3600 seconds), it would accelerate by $2.6\ \text{km/h/s} \cdot 1\ \text{h} = 9360\ \text{km/h}$.

From the relationship $v = a \cdot t$, we can now express the time it takes for the train to accelerate to its maximum speed. For uniformly accelerated motion starting from rest, the following applies

$$s = \frac{1}{2}at^2 = \frac{1}{2}v_{\max}t. \tag{9.1}$$

When we substitute $\frac{v_{\max}}{a}$ for t, we find that the distance travelled by the train before it accelerates to its maximum speed is

$$s = \frac{1}{2} \cdot v_{\max} \cdot \frac{v_{\max}}{a} = \frac{v_{\max}^2}{2a}. \tag{9.2}$$

By substituting values in the correct units for both variables, we obtain the distance 5.47 km.

10 The resulting force F_t acting on a fully submerged ball is the resultant of buoyant and gravitational forces, i.e.

$$F_t = F_b - F_g = V\rho_v g - V\rho_d g = (\rho_v - \rho_d)\frac{4}{3}\pi r^3 g. \tag{10.1}$$

The work done by the water sprite is determined by the formula $W = F_t \cdot h$, which gives us the answer

$$W = (\rho_v - \rho_d)\frac{4}{3}\pi r^3 g \cdot h \doteq 2.5\ \mathrm{J}. \tag{10.2}$$

11 By the time the tram reaches the crossing, the sound of the bell must have reached the driver, and the car must have had enough time to move out of the way. To find the minimum distance needed for this to happen, the time for the tram to reach the crossing (t_{tram}) must equal the sum of the time it takes the sound to reach the car (t_{sound}) and the time the car needs to leave (t).

Let s be the required distance between the tram and the crossing. The values of times can be expressed as

$$t_{\mathrm{tram}} = \frac{s}{v} \qquad \text{and} \qquad t_{\mathrm{sound}} = \frac{s}{c}. \tag{11.1}$$

Substituting these expressions into the time equation gives

$$t_{\mathrm{tram}} = t_{\mathrm{sound}} + t \quad \Leftrightarrow \quad \frac{s}{v} = \frac{s}{c} + t, \tag{11.2}$$

from which we can express the distance

$$s = \frac{c \cdot t \cdot v}{c - v}. \tag{11.3}$$

12 Let's start by determining how large area we actually want to cover. We approximate the Earth as a sphere with radius $R_\oplus = 6.378 \cdot 10^6$ m. Its surface area can be expressed as

$$S = 4 \cdot \pi \cdot r^2 \approx 5.1 \cdot 10^{14}\ \mathrm{m}^2. \tag{12.1}$$

From the task description, we know that the area of `A0` paper is 1 m^2 and that the size decreases by half with each increasing number. For example, the most common A4 paper has an area of $\frac{1}{2^4}\ \mathrm{m}^2 = \frac{1}{16}\ \mathrm{m}^2$.

However, we could also continue in the opposite direction towards negative numbers. A−1 paper would therefore have an area of 2 m^2, and each subsequent one would be double that… until we exceed the value from equation 12.1. For such cases, we can use a handy function called the logarithm with base 2. We can calculate this on a calculator[2].

We should not forget that we want to round it up, since we need the *smallest sufficiently large* paper. The result will therefore be

$$-\lceil N \rceil = -\left\lceil \log_2 \left(5.1 \cdot 10^{14}\right) \right\rceil = -49, \tag{12.2}$$

thus we are looking for paper size A−49.

[2] If our calculator cannot do this directly, we can use the relationship $\log_y x = \frac{\log x}{\log y}$.

13 The smoke does not move relative to the air – that is, it moves with the air at the wind speed v relative to the ground. By the time the fighter jet reaches the end of the grandstand, the beginning of the smoke trail is carried in the opposite direction and travels some distance. This time is

$$t = \frac{s}{u} \tag{13.1}$$

and during this time, the foremostly released smoke travels

$$s_d = vt = v\frac{s}{u}, \tag{13.2}$$

thus the length of the smoke trail is

$$s + s_d = s + v\frac{s}{u} = s\left(1 + \frac{v}{u}\right). \tag{13.3}$$

14 Since the prism has a higher refractive index than air, the rays will refract toward the normal when they hit it. To prevent any rays from passing to the other side of the prism, the rays must refract at least parallel to the wall of the prism behind which the guard is sitting, or they must be directed more steeply toward the lower base. From the figure, we can see that for this to happen, they must refract at an angle of at most 10° from the normal.

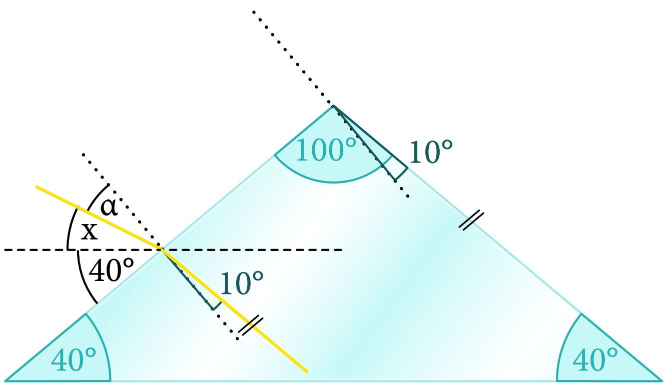


Figure 14.1: *Overview of relevant angles in the prism*

From this, we can use Snell's law to determine the angle at which the rays must strike,

$$n_1 \sin\alpha = n_2 \sin\beta \quad \Rightarrow \quad \alpha = \arcsin(2.4 \cdot \sin 10^\circ). \tag{14.1}$$

Since the prism is isosceles, its faces are at an angle of 40° to the horizontal direction. So when we look at the angles between the normal and the interface, we see that $40^\circ + \gamma + \alpha = 90^\circ$, where γ is the angle at which the thief must tilt the flashlight. This angle is therefore

$$\gamma = 50^\circ - \alpha = 50^\circ - \arcsin(2.4 \cdot \sin 10^\circ) \doteq 25.4^\circ. \tag{14.2}$$

15 First of all, we realize that the elevator's speed is irrelevant here. If Kai held the dumbell motionless, it would exert a force of $m_c g$ on him. However, he lets it fall with acceleration a, so it is sufficient for him to exert a force of $m_c(g-a)$. In addition to this, the elevator's cable also needs to carry Kai and the elevator itself, so it will also need to exert $m_k g$ and $m_v g$, so in total it must exert a force of

$$\begin{aligned} F &= m_k g + m_v g + m_c(g-a) \\ &= 60\ \text{kg}\cdot 10\ \text{m/s}^2 + 250\ \text{kg}\cdot 10\ \text{m/s}^2 + 20\ \text{kg}\cdot\left(10\ \text{m/s}^2 - 5\ \text{m/s}^2\right) = 3200\ \text{N} \doteq 3.2\ \text{kN}. \end{aligned} \tag{15.1}$$

16 In all three cases, it is a vertical throw in the same gravitational field, differing only in the initial conditions. Therefore, it will suffice to know the formulas for the trajectory and duration of uniformly accelerated motion,

$$s = \frac{1}{2}gt^2 \quad \Leftrightarrow \quad t = \sqrt{\frac{2s}{g}}. \tag{16.1}$$

Note that the second half of the upward throw is exactly the same as the entire downward throw: at the level of the window, the egg will definitely reach the same speed v_0 as the speed with which Sam threw it. The first half of the upward throw therefore lasts 2 s and during that time, the speed of the egg changes by $2v_0 = g\cdot 2$ s. Sam is therefore clearly throwing at a speed of $v_0 = 10$ m/s, which means that the egg will fly to a height of

$$h = \frac{v_0^2}{2g} \approx 5\ \text{m} \tag{16.2}$$

above the level of the window. At the same time, we know that at the highest point of this movement, the egg will stop, and from that moment on, it will be the simplest uniformly accelerated movement with zero initial velocity. The fall from the highest point will take

$$\tau = t_\uparrow - \frac{t_\uparrow - t_\downarrow}{2} = \frac{t_\uparrow + t_\downarrow}{2} = 3\ \text{s}, \tag{16.3}$$

from which we know that $H = \frac{g\tau^2}{2} = 45$ m. Sam's window is therefore at a height of $H - h = 40$ m above the ground. Free fall from this height takes

$$\sqrt{\frac{2(H-h)}{g}} = \sqrt{8}\ \text{s} \doteq 2.83\ \text{s}. \tag{16.4}$$

17 First, we draw the free-body diagram. All we need is Newton's second law, $F = ma$.

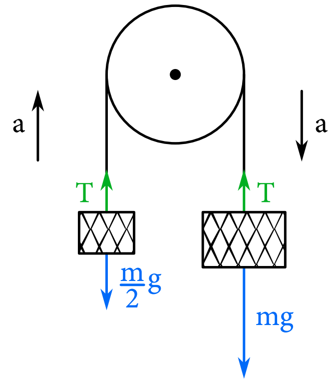


Figure 17.1: *Two weights on a pulley*

Gravity acts on both weights: $F_1 = mg$ and $F_2 = \frac{m}{2}g$. Because F_1 and F_2 act on the rope as well, it is being stretched, and therefore exerts a tension force on both ends, in the direction opposite to gravity. These two forces are labeled T_1 and T_2 in the free-body diagram. Since the rope and pulley are massless, T_1 must equal T_2; otherwise, it would cause an infinite acceleration of the rope itself. Because of this, we denote both T_1 and T_2 simply as T.

Additionally, we can assume that both weights have the same acceleration a, with the heavier one accelerating downwards. Now the only thing left to do is to write the equations of motion for both weights and solve for a.

$$\frac{m}{2}a = -\frac{m}{2}g + T \qquad \text{a} \qquad ma = mg - T. \tag{17.1}$$

By adding the equations, we can eliminate T and solve for a,

$$a = \frac{m - \frac{m}{2}}{m + \frac{m}{2}}g = \frac{1}{3}g. \tag{17.2}$$

18 The numbers are powers of two, so we have to assess the validity of every statement separately.

Increasing the orbital speed (1)

Intuitive guess would say that if we increase the orbital speed, the orbital period should decrease. But we must not forget that if we change the orbital speed, the shape of the orbit changes too. To find out whether this statement is true or false, we can look at a situation in which we increase the speed so much, so that the orbiting body acquires escape velocity. In that case, the orbital period would become effectively infinite, so it would definitely increase. Therefore, this statement is **false**.

Orbital speed in perihelion (2)

It follows directly from the second Kepler's law of planetary motion that the closer the Earth is to the Sun, the faster is travels. Therefore, this statement is **true**.

Winter (4)

Let us realise that when winter occurs in the northern hemisphere, it is summer in the southern hemisphere. If this statement were true, winter would have to occur in both hemispheres at the same time. Therefore, this statement is **false**. In fact, when the Earth is in its closest point to the Sun in the beginning of January, there is freezing winter in the northern hemisphere. Or maybe not so freezing.

Circular orbit and the first cosmic velocity (8)

The centripetal force causing a body to orbit at a circular orbit is the gravitational force, which depends on the radius r as $\frac{1}{r^2}$. However, the centripetal force is proportional to $\frac{v^2}{r}$, where v is the orbital velocity. This means that the circular velocity changes with height h above the Earth's surface as $v \propto \frac{1}{R+h}$. Therefore, this statement is **false**.

Geostationary satellites (16)

Geostationary satellites are satellites that are orbiting the Earth, but they are always above the same point of the Earth's surface. This is achieved by putting the satellite in such height, so that its angular velocity at which it orbits the Earth is the same as the anglar velocity of the Earth's rotation. If a satellite remains above the same point on the surface, it is orbiting in the direction of parallels. Should the satellite, however, orbit above any other parallel than the equator, the gravity acting towards the Earth's centre would attract it to the equator, making it impossible for the satellite to stay above its parallel. Hence the geostationary satellites can really exist only above the equator. Therefore, this statement is **true**.

Elliptic trajectories (32)

The lighter body is in an elliptic orbit if it is gravitationally bound to the much heavier body (and by chance does not have a circular velocity). However, if its speed is too high, it can escape to infinity, and in that case it is travelling on a hyperbolic or parabolic orbit. Therefore, this statement is **false**.

By summing the numbers of the true statements, we get the answer 18.

19 A good first step is to note that all edges and all pairs of vertices in a regular tetrahedron are equivalent. It therefore does not matter how we rotate it or which two vertices we connect the ohmmeter at the beginning to, the result will not change; we will measure different resistances only after removing the resistor.

Therefore, we can solve an equivalent but slightly simpler task: first, we remove any resistor and only then connect the ohmmeter terminals to any two vertices.

We should immediately see that we need to solve[3] only three different cases from Figure 19.1:

[3] We will use the abbreviated notation for the resistance of two parallel branches, $a \parallel b \equiv \frac{1}{\frac{1}{a}+\frac{1}{b}}$.

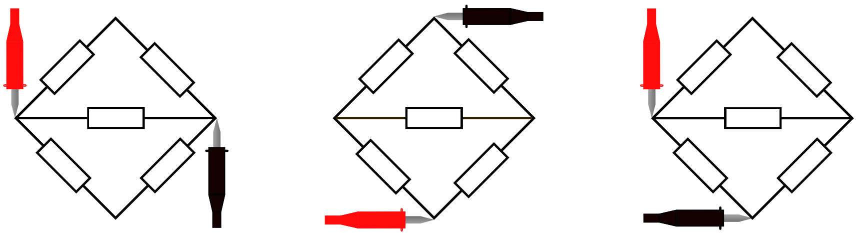


Figure 19.1: *Three possible resistance measurements on a tetrahedron without one resistor*

- If we connect it to two vertices to which the removed resistor was not connected, we are left with three parallel branches with one, two, and two resistors, for which we can calculate the total resistance directly as

$$R_1 = R \parallel (R + R) \parallel (R + R) = R \parallel 2R \parallel 2R = \frac{1}{2}R. \tag{19.1}$$

- If we connect the terminals to two vertices between which there is no longer any resistor, we get a pleasantly simple connection with two parallel branches with two resistors. Since the connection is now completely symmetrical, there can be no voltage between the ends of the last resistor, and where is no voltage, there is no current. Therefore, its resistance is

$$R_2 = (R + R) \parallel (R + R) = 2R \parallel 2R = R. \tag{19.2}$$

- Finally, if we connect the terminals to one untouched vertex and one vertex without a resistor, we are left with a series-parallel connection with a resistance of

$$R_3 = R \parallel (R + (R \parallel 2R)) = R \parallel \left(R + \frac{2}{3}R\right) = R \parallel \frac{5}{3}R = \frac{5}{8}R. \tag{19.3}$$

These three values are unique, so the answer is the set $\left\{\frac{1}{2}R; \frac{5}{8}R; R\right\}$.

20 To make the sorting easier, we will divide the letters into several groups. We should immediately suspect that many letters will not stand on their own.

For example, `J` and `C` stand on a single point, but their centre of gravity is not directly above this point, so they will immediately tip over. Similarly, `P` will also immediately tip over. Although it stands on a base of finite length, its centre of gravity is also definitely further to the right, so it will tip over the edge.

The second group includes the letters `O` and `S`. These will not fall on their own, because their centres of gravity are directly above the points where they touch the base; however, it is enough to tap them with any small impulse, and they will tip over.

The letters `I` and `Y` form another category. Since the thickness of the rods is non-zero, they stand stably; however it is much smaller than the other dimensions of the sticks, and a small (but unlike `O` and `S`, *finitely* small) impulse is enough to knock them over. Of the two, `Y` has significantly more mass, but its centre of gravity is slightly higher. Let's calculate which of these two letters is more stable.

Consider a letter with mass m, whose centre of gravity is at height H and for which the edge over which it wants to fall is ε away from the centre of gravity in the horizontal direction. When the letter is tipped over the edge, its potential energy increases by

$$\Delta E_{\text{pot}} = mg\left(\sqrt{H^2+\varepsilon^2} - H\right). \tag{20.1}$$

This is the work by which we quantify the stability of the letter. For $\varepsilon \ll H$, the following applies

$$\sqrt{H^2+\varepsilon^2} \approx H\left(1+\frac{\varepsilon^2}{2H^2}\right) \quad \Rightarrow \quad W \approx \frac{mg\varepsilon^2}{2H}. \tag{20.2}$$

The ratio $\frac{m\varepsilon^2}{H}$ is therefore essential for stability. A letter with a higher ratio is more stable, and the weight of the letter is proportional to its size.

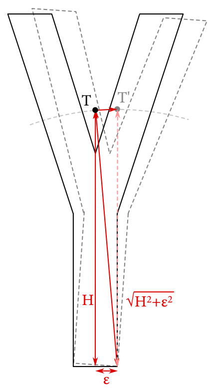


Figure 20.1: *Tipping the ypsilon. To tip the ypsilon, it must be lifted to the dotted position. We can see that its centre of gravity T′ is higher in this position than T in the original position.*

The letters `I` and `Y` have the same ε. The decisive factor is therefore the ratio $\frac{m}{H}$. Let one square of the square grid have a side of unit length and the bar of the letter of this length have unit mass. Then for `I` the height of the centre of gravity is 4 and its mass is 8, so the ratio $\frac{m}{H}$ is 2. For `Y`, H is approximately 4.76 and the mass is approximately 12.9, so the ratio determining stability is approximately 2.7. `Y` is therefore more stable than `I`.

Let's move on. The letters `N`, `A`, and `H` all have their centre of gravity exactly in the middle of the rectangle they describe and rotate around a point on the edge, which in all cases is located two squares from the centre horizontally and four squares vertically. Stability is therefore determined solely by their weight. By simply adding up the lengths of the sticks, we find that `N` has the greatest weight, followed by `H` and finally `A`.

There remains one last problematic solitaire – `B`. It rests at the bottom on a straight stick of length 2, which is connected to a semicircle with radius 2. If its centre of gravity were more than two squares away from its vertical stick, we would be done. However, an approximate calculation shows that its centre of gravity is slightly more than[4]

$$x = \frac{8 \cdot 0 + 6 \cdot 1 + 4\pi \cdot 3}{14 + 4\pi} \doteq 1.645. \tag{20.3}$$

This means that the centre of gravity of B is less than 0.355 horizontally away from the edge over which we want to tip it, which is definitely less than in the case of the letters from the previous group, but more than for I and Y, so it will lie between them in the stability series.^[For B, ε is sufficiently smaller than the height of the centre of gravity, so the same strategy can be used to examine stability as for I and Y. However, we do not need to calculate it accurately, because B has a much larger ε, which morover appears squared in the stability ratio, thus B will definitely be more stable.

The answer is therefore N, H, A, B, Y, I. The remaining letters will either fall down on their own, or very little effort is needed to knock them over.

21 We know that the inflow into the tank Q is constant. However, the speed of outflow through a small hole at the bottom of the tank will also depend on the hydrostatic pressure, which in turn depends on the height of the level in the tank h. Finding the height of the water column as a function of time is probably too difficult for us.[5] Fortunately, we do not need to, because the question in the problem statement is about a single parameter.

The asymptote at $t \to \infty$ determines the maximum water level in the tank $h_{\max} = 80$ cm – that is, the height at which the hydrostatic pressure at the bottom is sufficient for the water to flow out as fast as it flows in.

Then we can see the dotted tangent at $t = 0$. At that point, the water level is still zero, so the hydrostatic pressure is also zero and the water does not flow out at all, as if there were no hole in the bottom. Then the tank would obviously fill up in time $\tau = 1$ h. We can therefore express the inflow as

$$Q = \frac{h_{\max} \cdot S}{\tau}. \tag{21.1}$$

From Bernoulli's equation, we can quickly determine the outflow velocity as a function of the height of the water surface in the tank,

$$\rho \frac{v(h)^2}{2} = \rho g h \quad \Rightarrow \quad v(h) = \sqrt{2gh}. \tag{21.2}$$

We know that at a level $h_{\max} = 80$ cm, the inflow will be equal to the outflow, which is given by the product of the outflow velocity and the unknown area of the hole A. Then

$$Av = A\sqrt{2gh_{\max}} = Q, \tag{21.3}$$

and after substituting for Q from equation 21.1, we get the result

$$A = \frac{Q}{\sqrt{2gh_{\max}}} = \frac{S}{\tau}\sqrt{\frac{h_{\max}}{2g}} = 56\ \text{mm}^2. \tag{21.4}$$

[4] This is only a lower estimate, because the centre of gravity of the semicircle is a little closer to its edge, so its coordinate is slightly greater than 3.

[5] It leads to a Chini differential equation $\frac{\mathrm{d}h}{\mathrm{d}t} = \alpha + \beta\sqrt{h}$, which is formulated in terms of Lambert W function, and which we certainly do not want to solve.

22 We can immediately calculate that there is exactly 100 ml of yogurt in the milk, since this is 10 % of 1 l.

When the milk settles, there is only water at the top, hence the concentration of yogurt is 0 % upthere. For the average concentration to be 10 %, the concentration at the bottom must be 20 %, and thus increases by 2 % for every 100 ml. The top 200 ml therefore has an average concentration of 2 %, which is a total of 4 ml of yogurt.

The remaining volume in the bottle is 800 ml and the remaining volume of yogurt is 96 ml. When shaken, the concentration homogenizes, so it will be $\frac{96\ \text{ml}}{800\ \text{ml}} = 12\ \%$. From this, Mike poured 200 ml, hence 24 ml of yogurt.

Finally, we just add up the volume of yogurt in the glass,

$$4\ \text{ml} + 24\ \text{ml} = 28\ \text{ml}, \tag{22.1}$$

and calculate the total concentration as the ratio of the volume of the yogurt component to the total volume, i.e.

$$\frac{28\ \text{ml}}{400\ \text{ml}} = 7\ \%. \tag{22.2}$$

23 Wording of the problem implies that the log is inclined with respect to the horizontal direction by an angle α. Weight Mg acts vertically downwards at the centre of the log, Matthew acts perpendicular to its surface with a force F_M at its upper end, and Jacob acts with a force F_K at the lower end. This force can be broken down into a horizontal component F_{Kx} and a vertical component F_{Ky}.

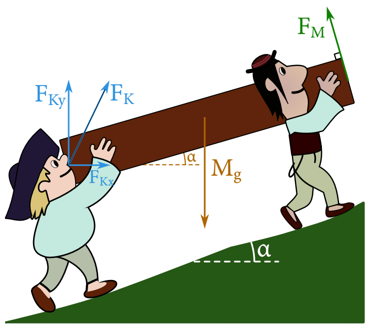


Figure 23.1: *Breakdown of the forces exerted on the load by Matthew and Jacob*

In the equilibrium position, we can write

- for forces in the vertical direction $Mg = F_M \cos\alpha + F_{Ky}$;
- for forces in the horizontal direction $F_M \sin\alpha = F_{Kx}$;
- for moments of forces relative to the lower end of the load $Mg\frac{L}{2}\cos\alpha = F_M L$.

Solving this system of equations, we obtain

$$F_M = \frac{1}{2} Mg \cos\alpha,$$
$$F_{Kx} = \frac{1}{2} Mg \sin\alpha \cos\alpha, \qquad (23.1)$$
$$F_{Ky} = \frac{1}{2} Mg + \frac{1}{2} Mg \sin^2\alpha.$$

The force exerted by Jacob is

$$F_K = \sqrt{F_{Kx}^2 + F_{Ky}^2} = \frac{1}{2} Mg\sqrt{3\sin^2\alpha + 1}, \qquad (23.2)$$

and the desired ratio of forces is finally

$$\frac{F_M}{F_K} = \frac{\cos\alpha}{\sqrt{3\sin^2\alpha + 1}}. \qquad (23.3)$$

24 We don't need to worry about longitude: a day is much shorter than a year, and the position of the Sun changes much faster in the east-west direction than in the north-south direction. During a single day, we can therefore consider the Sun's latitude to be constant and it is sufficient to look at the latitude.

The tilt of the Earth's axis determines how far north or south the Sun can travel during the year. These parallels are called the Tropic of Cancer (June solstice) and the Tropic of Capricorn (December solstice). At the same time, in December, when the Sun is furthest south, the northern polar region does not see the Sun at all. The southernmost parallel where this can occur is called the Arctic Circle. The same thing happens in the south, only in reverse and six months later. The latitude of the tropics is $\varepsilon_\oplus = \pm 23.44°$ and the latitude of the polar circles is $\pm(90° - \varepsilon_\oplus) \approx \pm 66.56°$.

The whole situation can be seen in Figure 24.1.

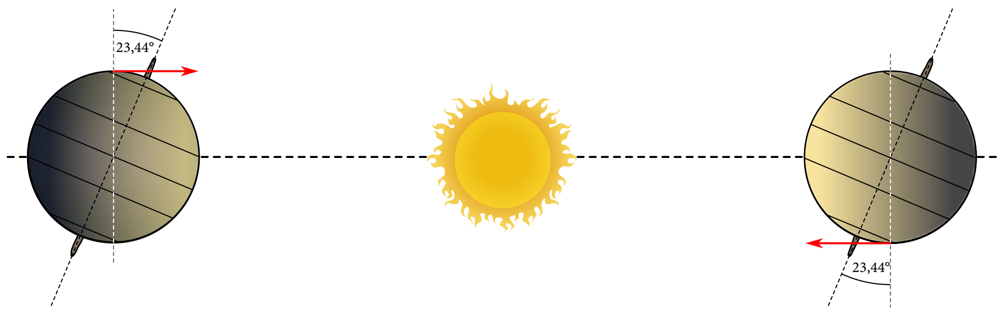


Figure 24.1: *Relative position of the Earth and Sun during the June and December solstices*

These circles divide the Earth's surface into five regions where the position of the Sun in the sky behaves differently throughout the year:

- In the southern polar region, at least once a year, there is a polar day when the Sun does not set at all. As it orbits around the horizon, it necessarily crosses the northern direction.
- Between the Tropic of Capricorn and the Antarctic Circle, we can see the Sun in the north every day, regardless of the season, at solar noon.
- In the tropics, i.e. between the tropics, the Sun can be at solar noon in the north or south depending on the season. We are interested in the June solstice, when the Sun reaches its northernmost position, and therefore will be in the north at noon.
- In the northern temperate zone, i.e. between the Tropic of Cancer and the Arctic Circle, we see the Sun at solar noon only in the south. At midnight, it is located to the north, but it is below the horizon, so we cannot see it.
- Finally, inside the Arctic Circle, the Sun does not set at all during the June solstice. Again, during its orbit, it necessarily crosses the north direction, but this does not happen at noon, but at midnight. The only exception is the North Pole itself, where all directions are "south"; however, this is only a single point and it does not contribute to the area.

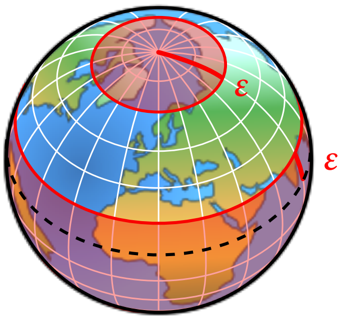


Figure 24.2: *Earth with marked areas where the Sun can be seen in the north*

It remains to find the sum of the areas of all regions south of the Tropic of Cancer $S_\downarrow$ and the area north of the Arctic Circle $S_\uparrow$, which are marked in the figure 24.2. We can calculate this, for example, using the formula for the surface area of a spherical cap. For that, we will need the heights of both areas: for the polar cap, it is $h_\uparrow = R(1 - \cos\varepsilon)$, and for the area south of the Tropic of Cancer, $h_\downarrow = R(1 + \sin\varepsilon)$. Hence together

$$S = S_\downarrow + S_\uparrow = 2\pi R h_\downarrow + 2\pi R h_\uparrow = 2\pi R(h_\downarrow + h_\uparrow) = 2\pi R^2(1 - \cos\varepsilon + 1 + \sin\varepsilon) = 2\pi R^2(2 + \sin\varepsilon - \cos\varepsilon), \quad (24.1)$$

and since we are interested in the ratio to the surface of the entire Earth $S_\oplus$, the answer is

$$\frac{S}{S_\oplus} = \frac{2\pi R^2(2 + \sin\varepsilon - \cos\varepsilon)}{4\pi R^2} = \frac{2 + \sin\varepsilon - \cos\varepsilon}{2} \doteq 0.74 = 74\ \%. \quad (24.2)$$

25 Dan, like any good archer, thinks about where to aim. He knows the distance to the target $d = 70$ m and the speed $v_0 = 50$ m/s. To know where to aim, he draws a figure of the system right before shooting.

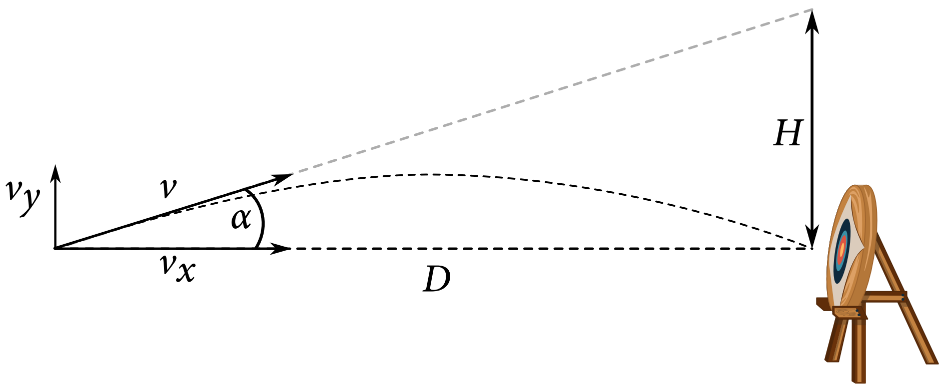


Figure 25.1: *Shot at a target*

First, he notices that the point he is aiming at is at a height H above the center of the target, and he can express the relationship for it as

$$\tan\alpha = \frac{H}{D}. \tag{25.1}$$

All that remains is to express $\tan\alpha$, which also appears in the decomposition of the initial velocity into the x- and y-directions

$$\tan\alpha = \frac{v_y}{v_x}. \tag{25.2}$$

In the y direction, only the gravitational force acts on the arrow, so in this direction, we have

$$y = v_y t - \frac{1}{2} g t^2, \tag{25.3}$$

where we are interested in the impact of the arrow on the target plane, i.e. $y = 0$. This occurs at time

$$t = \frac{2v_y}{g} = \frac{2v\sin\alpha}{g}. \tag{25.4}$$

No force acts on the arrow in the x direction, so between time t and distance D, the following applies

$$D = v_x t = v\cos\alpha \frac{2v\sin\alpha}{g}. \tag{25.5}$$

Since H and D are related through the tangent of the angle, we use trigonometric identities

$$\sin\alpha = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}, \qquad \cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}}. \tag{25.6}$$

After substituting into the equation for the range, we get

$$D = \frac{2v^2}{g}\frac{\tan\alpha}{1+\tan^2\alpha}. \tag{25.7}$$

We substitute H/D for $\tan\alpha$ according to equation 25.1, and after manipulations, we get a quadratic equation

$$H^2 - \frac{2v^2}{g}H + D^2 = 0, \tag{25.8}$$

the solution of which is[6]

$$H = \frac{v^2}{g} - \sqrt{\frac{v^4}{g^2} - D^2} \approx 10\ \text{m}. \tag{25.9}$$

26 Let's start by quantifying the defining units of the gulp-belch-wink system. We won't have much trouble with volume and time. We know that

$$V = 1\ \text{gulp} = \frac{1\ \text{pt}}{20} \approx \frac{473\ \text{ml}}{20} = 23.65\ \text{ml}, \tag{26.1}$$

and $t = 1$ wink $= 0.5$ s. Determining the sound intensity will be slightly more difficult. Let I be the sound intensity 1 belch. We know that it is defined as the sound intensity corresponding to the sound intensity level 100 dB, i.e. 10 B. Based on the definition of sound intensity level $L = \log\frac{I}{I_0}$, where $I_0 = 1\ \text{pW/m}^2$, we can write $10\ \text{B} = \log\frac{I}{I_0}$, from which

$$I = 10^{10} I_0 = 10^{10} \cdot 1\ \text{pW/m}^2 = 10^{-2}\ \text{W/m}^2. \tag{26.2}$$

Now we need to combine the quantities V, t and I to obtain something with the dimension kg. So we are looking for an expression of the form $m = V^\alpha \cdot t^\beta \cdot I^\gamma$. It should be true that $[V]^\alpha \cdot [t]^\beta \cdot [I]^\gamma$, where $[A]$ denotes the dimension (i.e., unit) of the quantity A, gives the dimension of mass. We obtain

$$[m] = \left(\text{m}^3\right)^\alpha \cdot \text{s}^\beta \cdot \left(\text{kg/s}^3\right)^\gamma = \text{m}^{3\alpha} \cdot \text{kg}^\gamma \cdot \text{s}^{\beta-3\gamma} \stackrel{!}{=} \text{m}^0 \cdot \text{kg}^1 \cdot \text{s}^0. \tag{26.3}$$

From this, we can immediately see that $\alpha = 0$, $\gamma = 1$ and $\beta = 3$, thus

$$m = t^3 I = (0.5\ \text{s})^3 \cdot 10^{-2}\ \text{W/m}^2 = \frac{1}{800}\ \text{kg} = 1.25\ \text{g}. \tag{26.4}$$

Now nothing prevents us from calculating John's weight in his natural units,

$$M = 200\ \text{lb} \approx 90\,718\ \text{g} = 72\,575\ \text{belch} \cdot \text{wink}^3. \tag{26.5}$$

27 The greatest pressure in the large container can be achieved by gradually connecting all five smaller containers to it, because we need to maximize the difference between the pressure in the large container and the small connected container, as that is when the amount of gas transferred from the smaller container to the larger one is maximized. Let's look at a situation if the nitrogen man had only two small containers, labeled 1 and 2. The normal procedure is to gradually connect the small bottles 1 and 2 one after other to the large container. This leaves pressures p_1 and p_2 in the small bottles, while pressure p_2 remains in the large container. An alternative procedure is to connect the small container 1, then equalize the pressures between containers 1 and 2, and then gradually connect containers 1 and 2. In this way, we connect the small containers to the large one several times, leaving pressure p_1' in container 1, which is greater than p_1, and thus

[6] We chose the smaller root.

the resulting pressure p_2' in the large container (and also in the small container 2) is less than p_2. Therefore, it is not worthwhile to equalize the pressures between the small containers.

From the equation of state of an ideal gas, we know that the amount of gas in both the small and large containers before we connect them can be determined from the state equation of the gas as $n = \frac{pV}{RT}$. We know that the total amount of substance n before and after we connect the bottles must be the same, i.e., the sum of n in the large and small bottles before connection must be equal to the total n after connection. If we denote the quantities describing the small bottle by the subscripts s, the large container by l, and the total state after connection by t, then

$$\frac{p_l V_l}{RT} + \frac{p_s V_s}{RT} = \frac{p_t V_t}{RT}. \tag{27.1}$$

Since the temperature and universal gas constant R remain constant throughout the process, we can express the total (equalized) pressure after connecting the cylinders as

$$p_t = \frac{p_l V_l + p_s V_s}{V_t}. \tag{27.2}$$

Before connecting the first small bottle, this pressure is zero, but after each subsequent connection, we increase this pressure – and the new resulting pressure in the connected bottles naturally depends on what the pressure in the large bottle was before.

We therefore perform the calculation for p_t five times; the first time we substitute 0 for p_l, and each subsequent time we substitute p_t, which we have obtained the most recently. As the last fifth p_t, we get exactly $\frac{23255}{7776}$ MPa, or approximately 3 MPa.

28 The plate with platter must be spun at an angular velocity of

$$\omega = \frac{100}{3} \cdot \frac{2\pi\ \text{rad}}{60\ \text{s}} = \frac{10}{9}\pi\ \text{rad/s}. \tag{28.1}$$

The rotational energy that must be supplied will therefore be $E_{\text{rot}} = \frac{1}{2}J\omega^2$. Since the moment of inertia of a solid disk is $J = \frac{1}{2}(m + M)r^2$, where r is the radius of the disk, we can substitute this for J and obtain the energy

$$E_{\text{rot}} = \frac{1}{2}\frac{1}{2}(m + M)r^2\omega^2. \tag{28.2}$$

We know what voltage and current the turntable requires, and we know that the power used to spin the record will be $P = UI$. We can now simply determine the time as the required energy divided by the power,

$$t = \frac{E_{\text{rot}}}{P} = \frac{\frac{1}{4}(m + M)\left(\frac{d}{2}\right)^2\omega^2}{UI} = \frac{(200\ \text{g} + 2\ \text{kg}) \cdot (15\ \text{cm})^2 \cdot (\frac{10}{9}\pi\ \text{rad/s})^2}{4 \cdot 5\ \text{V} \cdot 100\ \text{mA}} \approx 0.302\ \text{s}. \tag{28.3}$$

29 Before we start calculating, we need to translate the problem into understandable language. Once we understand what the problem is, we can start.

In order for the entire system to be in equilibrium, the net force on each charge must be zero. Thanks to symmetry, we only need to look at the forces acting on a single charge located at the vertex of the cube. The only attractive force acting on it comes from the charge in the centre, the magnitude of which we are trying to find; the remaining charges act with repulsive forces – three charges at a distance equal to the cube's edge, three charges at a distance equal to the face diagonal, and finally one charge at a distance equal to the space diagonal. Let us denote the edge length of the cube as a. The face diagonal then has length $\sqrt{2}a$ and the space diagonal $\sqrt{3}a$.

Each force acting on the chosen vertex charge can be decomposed into two components - one in the direction of the resultant force, i.e., along the space diagonal, and the other perpendicular to it. The components perpendicular to the space diagonal cancel each other out due to symmetry, and the components along the space diagonal add up to form the resultant force. Therefore, we are interested in the cosine of the angle between the space diagonal and the cube's edge, and between the space diagonal and the face diagonal.

Stop reading for a moment, this is a mandatory vibe check.

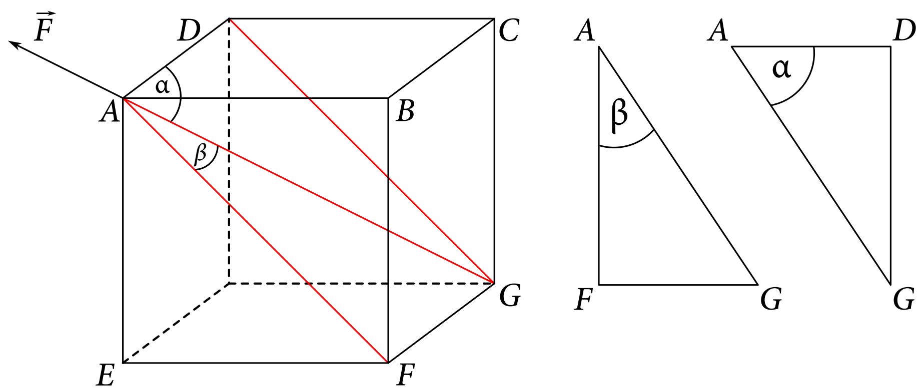


Figure 29.1: *Skibidicube*

Ok, vibe check passed. Next, let us calculate the force from the charge located at a distance equal to the cube's edge. The cosine of the required angle can be found from the right triangle ADG as

$$\cos\alpha = \frac{1}{\sqrt{3}}. \tag{29.1}$$

The force will therefore be

$$F_\mathrm{h} = -k\frac{q^2}{\sqrt{3}a^2}. \tag{29.2}$$

Next, let us look at the force from the charge located at a distance equal to the face diagonal. The cosine of the required angle can be found from the right triangle AFG as

$$\cos\beta = \frac{\sqrt{2}}{\sqrt{3}}. \tag{29.3}$$

The force will therefore be

$$F_\mathrm{s} = -k\frac{\sqrt{2}q^2}{2\sqrt{3}a^2}. \tag{29.4}$$

For the force from the charge located at a distance equal to the space diagonal, we don't need to deal with any angle, hence the force will be simply

$$F_\mathrm{t} = -k\frac{q^2}{3a^2}. \tag{29.5}$$

Lastly, we need to calculate the force from the charge in the center. This is the same situation as with the space diagonal, except the distance is halved. Therefore, the force will be

$$F_\mathrm{centre} = 4k\frac{qQ}{3a^2}. \tag{29.6}$$

The net force will therefore be

$$F = F_\mathrm{centre} - 3F_\mathrm{h} - 3F_\mathrm{s} - F_\mathrm{t} = 0. \tag{29.7}$$

Substituting in, we get

$$\begin{aligned} 4k\frac{qQ}{3a^2} &= -3k\frac{q^2}{\sqrt{3}a^2} - 3k\frac{\sqrt{2}q^2}{2\sqrt{3}a^2} - k\frac{q^2}{3a^2}, \\ 4\frac{Q}{3} &= -3\frac{q}{\sqrt{3}} - 3\frac{\sqrt{2}q}{2\sqrt{3}} + \frac{q}{3}, \\ Q &= -\frac{6\sqrt{3}+3\sqrt{6}+2}{8}q. \end{aligned} \tag{29.8}$$

The charge in the center must therefore be $Q \doteq -2.47\ \mathrm{C}$. GG EZ.

30 Let's write down the equations that describe projectile motion

$$\begin{aligned} y &= v\sin(\alpha)t - \frac{1}{2}at^2, \\ x &= v\cos(\alpha)t. \end{aligned} \tag{30.1}$$

On the Moon, the equations will stand, only the gravitational acceleration is different: if we substitute $a = 0.16g$, it should be

$$\begin{aligned} y_☾ &= v_☾\sin(\alpha)t_☾ - \frac{1}{2}0.16g{t_☾}^2, \\ x_☾ &= v_☾\cos(\alpha)t_☾. \end{aligned} \tag{30.2}$$

We cannot fake the position in the video. For the equations for x and y to remain the same, we need to adjust time and velocity. We know that

$$0.16{t_☾}^2 = t_\oplus^2 \quad \Rightarrow \quad t_☾ = \frac{1}{\sqrt{0.16}} t_\oplus = \frac{1}{0.4} t_\oplus = 2.5 t_\oplus. \tag{30.3}$$

From the equation for $x_☾$, we now see that

$$x_☾ = v_☾ \cos(\alpha) 2.5 t_\oplus, \tag{30.4}$$

thus $v_☾ = 0.4 v_0$ in order for equation 30.4 to equal the expression for x in equation 30.1. On the Moon, therefore, this whole projectile motion would last 2.5 times longer, which means that the Earth video must be played at a frame rate 2.5 times lower, i.e. 24 Hz. That means we simply move from the television frame rate to the movie frame rate.

Coincidence? We do not think so.

31 Let's consider a heat pump as a heat engine that draws energy from the electrical grid and transfers heat from Adam's cellar to Justine's cellar. The key to solving the problem is to understand how efficiency is defined.

In general, efficiency can be said to be the ratio of what we want to achieve to what we have to invest in it. In Justine's case, what we want to achieve is the amount of heat that the heat pump transfers to her cellar, Q_{out}, and what we have to invest is the amount of electrical energy supplied from the grid, which is equal to the work done on the working gas in the pump, W. The efficiency she observes is therefore $\eta_{\text{J}} = \frac{Q_{\text{out}}}{W} = 1500\ \%$.

In Adam's case, what we want to achieve is the amount of heat extracted by the heat pump from his cellar, Q_{in}, and what we have to invest is again the amount of electrical energy supplied, W, so the efficiency he observes is $\eta_{\text{G}} = \frac{Q_{\text{in}}}{W}$.

The heat pump extracts heat Q_{in} from Adam's cellar, and we supply it with energy W from the grid, and it transfers heat Q_{out} to Justina's cellar. The energy balance written for the heat pump is therefore $Q_{\text{in}} + W = Q_{\text{out}}$.

Using this relationship, the efficiency observed by Adam can be expressed as $\eta_{\text{G}} = \frac{Q_{\text{out}} - W}{W} = \eta_{\text{J}} - 1 = 1400\ \%$.

32 Let us look at the situation in the problem statement from the energy point of view. At the begenning of the day, the ISS has energy which is the sum of its potential and kinetic energy. After one day of experiencing the drag force, the ISS has lower energy because drag is acting against the direction of motion.

Let us denote energies at the beginning and at the end by 1 and 2 respectively. Then from the law of energy conservation follows that

$$E_{\text{p1}} + E_{\text{k1}} = E_{\text{p2}} + E_{\text{k2}} + W, \tag{32.1}$$

where E_{p} are potential energies, E_{k} are kinetic energies, and W is work which was done by drag force on the ISS.

The ISS is orbiting the Earth on a circular orbit, so we can easily express its orbital speed using the equality between the gravitational and centripetal force. For our purposes, it is handy to write the square of the speed[7]

$$v^2 = \frac{GM_\oplus}{r}, \tag{32.2}$$

where $M_\oplus$ si the Earth's mass, $r = R_\oplus + 400$ km is the distance of the ISS from the centre of the Earth, and $R_\oplus$ is self-explanatorily the radius of the Earth. Let us now denote the mass of the ISS m and its daily change in height Δr. Now we can rewrite the equation 32.1 as

$$-\frac{GM_\oplus m}{r} + \frac{1}{2}m\frac{GM_\oplus}{r} = -\frac{GM_\oplus m}{r-\Delta r} + \frac{1}{2}m\frac{GM_\oplus}{r-\Delta r} + W, \tag{32.3}$$

whence we express the work done by the drag force

$$W = \frac{GM_\oplus m}{2}\left(\frac{1}{r-\Delta r} - \frac{1}{r}\right). \tag{32.4}$$

The drag force F acts against the motion of the ISS, so after travelling the distance s, the work done by the drag force is $W = Fs$. What is the distance travelled by the ISS in one day? Since Δr is much smaller than the radius of the circular orbit, we do not have to calculate the length of the spiral, which is the true trajectory of the ISS. We can get a reasonably good result if we assume that the ISS is circulating all day at an altitude of $r - \Delta r/2$ at the speed that corresponds to this altitude according to the equation 32.2, hence

$$s = \sqrt{\frac{GM_\oplus}{r - \Delta r/2}}\,T, \tag{32.5}$$

where T is one day.

Now we simply substitute equations 32.4 and 32.5 into

$$F = \frac{W}{s}, \tag{32.6}$$

and for the given numeric values, we obtain $F \doteq 0.26$ N.

33 Let us consider rolling the roll to the maximum possible angle. In this position, the angle between the normal on the ISIC and the vertical direction is φ.

[7] Because we will substitute it into the expression for kinetic energy.

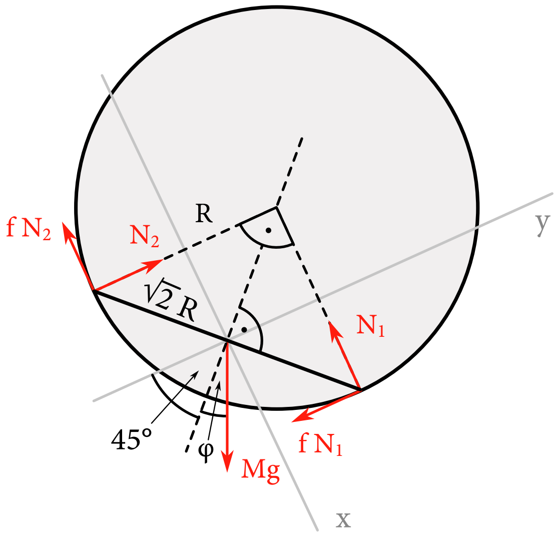


Figure 33.1: *Roll with forces acting on it*

Normal forces N_1 and N_2 act on the edges of the ISIC perpendicularly to the surface of the roll, and thus to the center of the roll. In the tangential direction, there are frictional forces, whose magnitudes in the limiting case are $f N_1$ and $f N_2$ respectively. The gravitational force Mg acts vertically downward at the centre of the ISIC.

The size of the ISIC is such that together with the lines of action of the normal forces, it forms a right-angled triangle with legs of length R and hypotenuse $\sqrt{2}R$. Among other things, this means that the forces N_1 and $f N_2$ have the same direction, that is perpendicular to the direction of the forces N_2 and $f N_1$. This encourages us to choose a coordinate system such that the coordinate axes are oriented in these two directions, i.e., rotated by 45° relative to the ISIC. Then the gravitational force decompose into components $Mg\cos\left(\frac{\pi}{4}-\varphi\right)$ and $Mg\cos\left(\frac{\pi}{4}+\varphi\right)$.

Now we can write the stability conditions:

- balance of forces in the x direction,

$$N_1 + f N_2 = Mg\cos\left(\frac{\pi}{4}-\varphi\right); \tag{33.1}$$

- balance of forces in the y direction,

$$N_2 = f N_1 + Mg\cos\left(\frac{\pi}{4}+\varphi\right); \tag{33.2}$$

- and equilibrium of torques with respect to the center of ISIC,

$$N_1 \cdot \frac{\sqrt{2}}{2}R \cdot \frac{\sqrt{2}}{2} = f N_1 \cdot \frac{\sqrt{2}}{2}R \cdot \frac{\sqrt{2}}{2} + N_2 \cdot \frac{\sqrt{2}}{2}R \cdot \frac{\sqrt{2}}{2} + f N_2 \cdot \frac{\sqrt{2}}{2}R \cdot \frac{\sqrt{2}}{2}, \tag{33.3}$$

therefore we can write

$$N_1 = f N_1 + N_2 + f N_2. \tag{33.4}$$

By eliminating the normal forces, we obtain

$$\underbrace{\frac{1+2f-f^2}{1-2f-f^2}}_{K} = \frac{\cos\left(\varphi - \frac{\pi}{4}\right)}{\cos\left(\varphi + \frac{\pi}{4}\right)} = \frac{\cos\varphi + \sin\varphi}{\cos\varphi - \sin\varphi}, \tag{33.5}$$

and from there

$$\tan\varphi = \frac{K-1}{K+1} = \frac{2f}{1-f^2}. \tag{33.6}$$

For $f = 0.3$, we obtain $\varphi \approx 33.4°$.

34 Let's start by writing the equation of motion of a car on a flat surface,

$$ma_- = fmg \quad \Rightarrow \quad f = \frac{a_-}{g}. \tag{34.1}$$

On an inclined plane, sloped at an angle α, this changes slightly to

$$ma = mg\sin\alpha + fmg\cos\alpha. \tag{34.2}$$

Compared to equation 34.1, there is one more term on the right-hand side – the acceleration of the car is still caused by the friction of the wheels on the road, but it diminishes with increasing the slope angle; and the motion is also affected by a component of the gravitational force parallel to the inclined plane. The acceleration is therefore

$$a = g\sin\alpha + fg\cos\alpha. \tag{34.3}$$

Analytical solution

The shortest acceleration time is obtained by maximising a. We therefore need to differentiate a with respect to angle α and find out when the derivative is equal to zero. We get

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}\alpha} &= g\cos\alpha - fg\sin\alpha \overset{!}{=} 0, \\ \cos\alpha &= f\sin\alpha, \\ \tan\alpha &= \frac{1}{f} = \frac{g}{a_-}. \end{aligned} \tag{34.4}$$

We will use trigonometric identities 25.6, substituting for $\sin\alpha$ and $\cos\alpha$ in 34.3. After minor rearrangement using 34.1, we get

$$a = \sqrt{a_-^2 + g^2}. \tag{34.5}$$

Now we just need to find the acceleration of the car on the flat surface and calculate the acceleration time downhill. We will use the fact that for uniformly accelerated motion, $v = at$, therefore

$$t = \frac{v}{\sqrt{a_-^2 + g^2}} = \frac{v}{\sqrt{\left(\frac{v}{t_-}\right)^2 + g^2}}, \tag{34.6}$$

where v is the maximum speed to which the car accelerates, and t_- is the acceleration time on the flat surface. We substitute the values and get $t \doteq 1.51$ s.

Numerical solution

If we do not want to differentiate, we can directly search for the maximum a from equation 34.3, for example, by a linear search within the interval using a calculator. If we draw the function from equation 34.3, it clearly has only one maximum, so it is sufficient to find the first decrease and, if necessary, refine the search step. This gives us the same intermediate result $\alpha \approx 0.595$ rad $\approx 34.1°$, from which we can calculate the acceleration and time.

35 When we look at the Earth and its shadow from a distance, its shadow is an infinite cylinder with a circular base.[8] For every fixed point on Earth's surface, the Sun sets when the Earth's rotation carries that point across the shadow boundary.

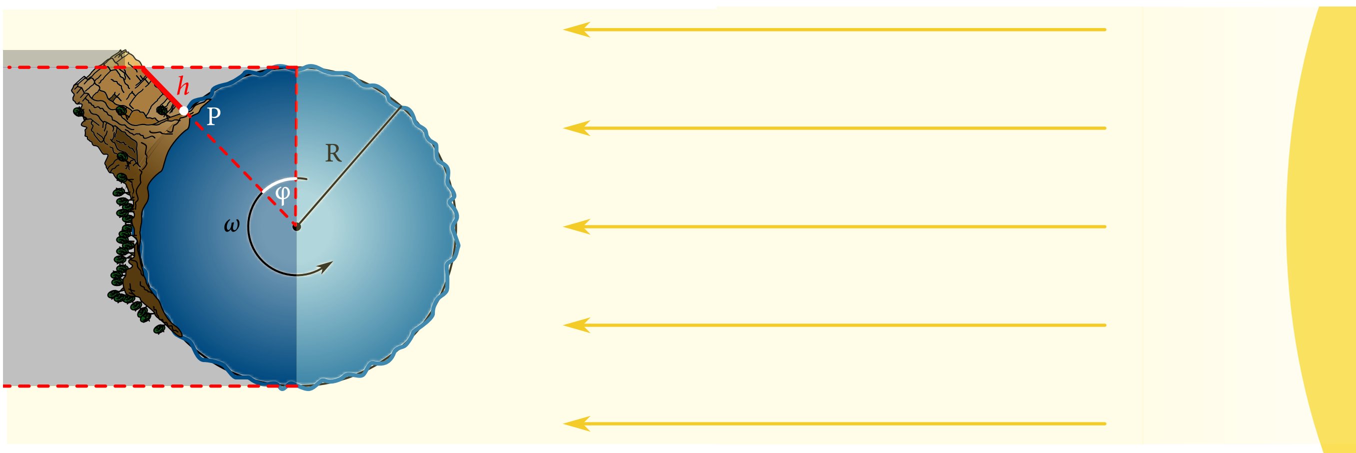


Figure 35.1: *Earth casting a shadow*

And not only on the surface – the same applies to points on the cliff. If the height of the cliff is negligible compared to the radius of the Earth – which it should be – then for any point on the cliff at height h, the Sun will set when point P reaches a depth of h inside the shadow, i.e., a distance of h from the cylinder's surface. The height of the shadow is therefore equal (in the case of a not very tall cliff) to the perpendicular distance of point P from the cylinder's surface.

[8] Considering its finite dimensions and distance from the Sun, it is actually a cone, but this is not important for our purposes.

And that's all. Point P moves along a circle with radius $R_\oplus$ at an angular velocity $\omega = \frac{2\pi}{1\,\mathrm{d}} = \frac{2\pi}{86\,400\,\mathrm{s}}$. The projection of such a motion onto a plane perpendicular to the sun's rays is harmonic motion with the same angular velocity, and its acceleration near the maximum deflection is given by

$$a = \omega^2 R_\oplus \approx \left(\frac{2\pi}{86\,400\,\mathrm{s}}\right)^2 \cdot 6.378 \cdot 10^6\,\mathrm{m} \doteq 0.034\,\mathrm{m/s^2}. \tag{35.1}$$

Lumberjack's solution

It is also possible to obtain the same result with brute force. Let's mark the central angle between the vertical cliff and the terminator φ. Then, from the right triangle containing this angle and enclosed by the shadow's envelope, we can write

$$R_\oplus + h(\varphi) = \frac{R_\oplus}{\cos\varphi}. \tag{35.2}$$

When the Earth rotates by an angle $\mathrm{d}\varphi$, the height to which the shadow moves increases by

$$\mathrm{d}h = \frac{R_\oplus}{\cos(\varphi + \mathrm{d}\varphi)} - \frac{R_\oplus}{\cos\varphi}. \tag{35.3}$$

For a low cliff, the angle φ is very small, so we can use the approximations $\cos x \approx 1 - \frac{x^2}{2}$ and $(1+x)^{-1} \approx 1 - x$. Then we obtain

$$\mathrm{d}h \approx R_\oplus\left[\frac{1}{1 - \frac{(\varphi+\mathrm{d}\varphi)^2}{2}} - \frac{1}{1 - \frac{\varphi^2}{2}}\right] \approx R_\oplus\left[1 + \frac{(\varphi + \mathrm{d}\varphi)^2}{2} - \left(1 + \frac{\varphi^2}{2}\right)\right] \approx R_\oplus \varphi\,\mathrm{d}\varphi. \tag{35.4}$$

Since the Earth rotates with a constant angular speed ω, $\varphi = \omega t$, and therefore

$$\mathrm{d}h \approx R_\oplus \omega t \omega\,\mathrm{d}t. \tag{35.5}$$

From this, we can immediately write the speed of the shadow's ascent along the cliff

$$v(t) = \frac{\mathrm{d}h}{\mathrm{d}t} \approx R_\oplus \omega^2 t \tag{35.6}$$

and therein we recognize the speed of uniformly accelerated motion with acceleration $a = \omega^2 R_\oplus$.

36 Firstly, Jacob thought about what do they mean by activity of natural and enriched uranium. He remembered the formula which relates the number of decayed atoms ΔN after time Δt and total number of atoms N in given time

$$\Delta N = -\lambda N\,\Delta t, \tag{36.1}$$

where λ is the decay constant. This equation leads to

$$N = N_0 e^{-\lambda t}, \tag{36.2}$$

known as exponential decay, where N_0 is the number of atoms in time $t = 0$.

The last thing we need to express is the half-life, after which half of the atoms decay. That means, that the number of atoms decreases to half of the original amount, $N = \frac{N_0}{2}$, which, after substituting into the exponential decay formula, gives the decay constant

$$\lambda = \frac{\ln 2}{T_{1/2}}. \tag{36.3}$$

Next, using the knowledge above, he can find the activity

$$A = -\frac{\Delta N}{\Delta t} = \lambda N = \frac{\ln 2}{T_{1/2}} N. \tag{36.4}$$

In order to calculate how much higher is the activity of enriched uranium compared to natural uranium, he needs to find the ratio $\frac{A_{\text{enriched}} - A_{\text{natural}}}{A_{\text{natural}}}$, where he can utilise his knowledge of the ratio of number of ^{235}U and ^{238}U atoms in both uranium species, and he knows their half-lives $T_{1/2}^{235} = 7 \cdot 10^8$ a a $T_{1/2}^{238} = 4.46 \cdot 10^9$ a, too. From the half-lives and the formula for activity, he knows that the decay constant λ of ^{235}U is k-times higher than the one of ^{238}U. Precisely, it is

$$k = \frac{T_{1/2}^{238}}{T_{1/2}^{235}} \doteq 6.371. \tag{36.5}$$

Then, if he denotes the ratio of ^{235}U in enriched uranium a, and b in natural uranium, he gets the expression for relative change of activity,

$$\frac{A_{\text{enriched}} - A_{\text{natural}}}{A_{\text{natural}}} = \frac{k\cancel{\lambda} a \cancel{N} + \cancel{\lambda}(1-a)\cancel{N}}{k\cancel{\lambda} b \cancel{N} + \cancel{\lambda}(1-b)\cancel{N}} - 1 = \frac{(a-b)(k-1)}{1+b(k-1)} \doteq \frac{(0.03 - 0.0072)\cdot(6.371-1)}{1+0.0072\cdot(6.371-1)} \doteq 11.8\ \%. \tag{36.6}$$

37 First, let us note that if the axis of rotation is perpendicular to the plane of the amulet, the way in which the individual cut-out squares are rotated does not affect the moment of inertia of the amulet. Let us call the first circle from which we cut out a square the first iteration. Now, when we inscribe another circle and cut out the inscribed square, we will call this the second iteration, and so on. The moment of inertia of a circle with radius r is $I_{\bigcirc} = \frac{1}{2}mr^2$ and that of a square is $I_{\square} = \frac{1}{6}ma^2$, where a is the side of the square.

Let's look at the first iteration of our problem. We have the radius of the circle of the first iteration $r_1 = r$ and an inscribed square with a side length of $a = \sqrt{2}r$, which we can easily see. We know that the diagonal of the square is $2r$. From the right-angled triangle defined by two adjacent vertices of the square and the centre of the circle, we see that

$$\cos 45^\circ = \frac{r}{a} \quad \Rightarrow \quad a = \sqrt{2}r. \tag{37.1}$$

The moment of inertia and the area of the first iteration are simply

$$\begin{aligned} I^{(1)} &= \frac{1}{2}\sigma S_{\bigcirc}^{(1)} r_1^2 - \frac{1}{6}\sigma S_{\square}^{(1)} a_1^2 = \frac{1}{2}\pi\sigma r_1^4 - \frac{4}{6}\sigma r_1^4 = \left(\frac{1}{2}\pi - \frac{2}{3}\right)\sigma r_1^4 \\ S^{(1)} &= S_{\bigcirc}^{(1)} - S_{\square}^{(1)} = \pi r_1^2 - a_1^2 = (\pi - 2) r_1^2. \end{aligned} \tag{37.2}$$

The moment of inertia $I^{(n)}$ and the area $S^{(n)}$ of the following iterations are calculated similarly, only for different values of r_n. All that remains is to determine the relationship $r_n = r_n(r, n)$. For the first iteration, $r_1 = r$. The second iteration is a circle inscribed in a square, and therefore

$$2r_2 = a_1 \quad \Rightarrow \quad r_2 = \frac{\sqrt{2}}{2} r_1 = \frac{\sqrt{2}}{2} r. \tag{37.3}$$

Similarly, the following applies

$$r_3 = \frac{\sqrt{2}}{2} r_2 = r\left(\frac{\sqrt{2}}{2}\right)^2, \tag{37.4}$$

and thus, in general, for r_n

$$r_n = \frac{\sqrt{2}}{2} r_{n-1} = r\left(\frac{\sqrt{2}}{2}\right)^{n-1}. \tag{37.5}$$

All that remains is to calculate the infinite series that appear,

$$\begin{aligned} S = \sum_{n=1}^{\infty} S^{(n)} = (\pi - 2)\sum_{n=1}^{\infty} r_n^2 = r^2(\pi-2)\sum_{n=1}^{\infty}\left(\frac{\sqrt{2}}{2}\right)^{2n-2} \\ = r^2(\pi-2)\frac{1}{1-\left(\frac{\sqrt{2}}{2}\right)^2} = 2r^2(\pi-2), \end{aligned} \tag{37.6}$$

and for the moment of inertia

$$\begin{aligned} I = \sum_{n=1}^{\infty} I^{(n)} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\pi - \frac{2}{3}\right)\sigma r_n^4 = \left(\frac{1}{2}\pi - \frac{2}{3}\right)\sigma r^4 \sum_{n=1}^{\infty}\left(\frac{\sqrt{2}}{2}\right)^{4n-4} \\ = \left(\frac{1}{2}\pi - \frac{2}{3}\right)\sigma r^4 \frac{1}{1-\left(\frac{\sqrt{2}}{2}\right)^4} = \frac{4}{3}\left(\frac{1}{2}\pi - \frac{2}{3}\right)\sigma r^4 = \frac{4}{3}\left(\frac{1}{2}\pi - \frac{2}{3}\right)\frac{m}{S} r^4, \end{aligned} \tag{37.7}$$

where we used the fact that the sum of a geometric series with a common ratio $0 < q < 1$ is $\sum_{n=1}^{\infty} q^n = \frac{1}{1-q}$. After substituting S into I, we get

$$I = \frac{2}{3}\frac{\frac{1}{2}\pi - \frac{2}{3}}{\pi - 2} m r^2 = \frac{3\pi - 4}{9\pi - 18} m r^2. \tag{37.8}$$

38 Let us start with the formula for the parabolic city,

$$h(r) = kr^2, \tag{38.1}$$

where r is the distance of a point on the paraboloid surface from the rotation axis, that is, from the city centre. We do not have to add a constant, since we are interested only in the height difference. For the highest point of the city, it holds that

$$H = kR^2, \tag{38.2}$$

where H is the sought height and R is given in the problem statement.

We still do not know the coefficient k. It can be found from the derivative of parabola in the point $[R;H]$. We know that the resulting apparent gravitational force must be perpendicular to the paraboloid surface. Derivative of the formula of parabola in the point $[R;H]$, $2kR$, is the slope of tangent to the surface of parabola. The slope of a line is found as tan $\frac{\Delta y}{\Delta x}$, where $\Delta y = a_c$ and $\Delta x = g_{♂}$, so

$$2kR = \frac{a_c}{g_{♂}} \qquad \Rightarrow \qquad k = \frac{a_c}{2g_{♂}R}. \tag{38.3}$$

After substituting k into the equation 38.2, we have

$$H = \frac{a_c R}{2g_{♂}}. \tag{38.4}$$

We still need to express a_c. We find it using the Pythagoean theorem,

$$a_c = \sqrt{g^2 - g_{♂}^2}. \tag{38.5}$$

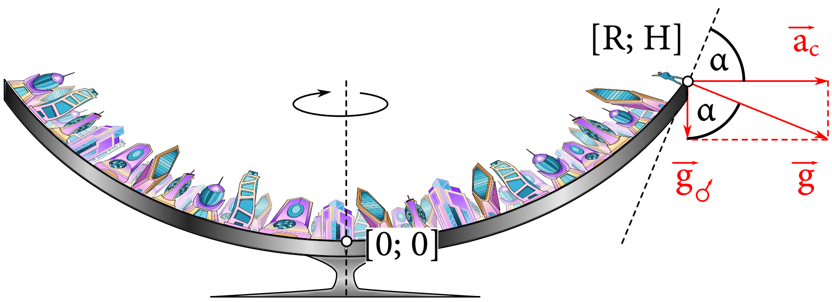


Figure 38.1: *Vertical cross section of the parabolic city*

The result is therefore

$$H = \frac{R\sqrt{g^2 - g_{♂}^2}}{2g_{♂}} \doteq 1296\ \text{m}. \tag{38.6}$$

39 Let us denote the volume of the lollipop V. Its shape is such that at any height, the cross section is an ellipse with a constant semi-major axis a and a semi-minor axis that scales linearly with the depth, as shown in figure 39.1.

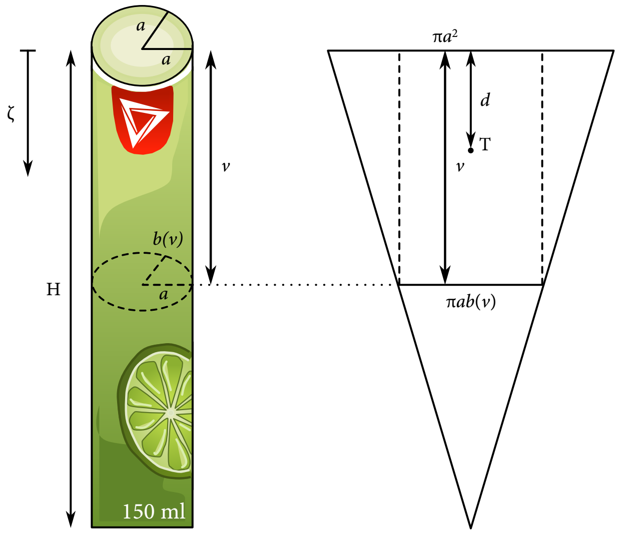


Figure 39.1: *Goemetry of the lollipop*

If we define a coordinate ζ with origin at the top and increasing downwards, b scales as

$$b(\zeta) = \frac{H - \zeta}{H} a, \tag{39.1}$$

and the area of the cross section at depth ζ is

$$S(\zeta) = \pi a b(\zeta) = \pi a^2 \frac{H - \zeta}{H}. \tag{39.2}$$

Notice that the cross section area decreases linearly with ζ. Therefore, a planar analogy to this "deformed cone" is a triangle, which has the same property. For calculating the centre of mass, it is therefore equivalent to a triangle with a base of length $\ell = S(0) = \pi a^2$ and the same altitude H. The area of the triangle corresponds to the volume of the lollipop V, so

$$V = A = \frac{1}{2}\ell H = \frac{1}{2}\pi a^2 H \qquad \Rightarrow \qquad H = \frac{2V}{\pi a^2}. \tag{39.3}$$

Now we are interested in the situation after George has eaten half of the lollipop, so we need to find the height of the part whose volume is half the original volume V. Let us denote the volume of the lollipop inside $[0; \zeta]$ as $\Theta(\zeta)$. This corresponds to the area of a trapezoid we can get by truncating the triangle at a distance of ζ from the base,

$$\Theta(\zeta) = \frac{\pi a^2 + \pi a b(\zeta)}{2}\zeta = \pi a^2 \frac{1 + \frac{H-\zeta}{H}}{2}\zeta = \pi a^2 \zeta \left(1 - \frac{\zeta}{2H}\right). \tag{39.4}$$

When the volume decreases to half its original value, we can write $\zeta = v$, and thus $\Theta(v) \stackrel{!}{=} \frac{V}{2}$. Therefore

$$\frac{1}{2}\pi a^2 H = \pi a^2 v\left(1 - \frac{v}{2H}\right) \quad \Rightarrow \quad v = \frac{2-\sqrt{2}}{2}H, \tag{39.5}$$

and George needs to push the lollipop that far out from its wrapper.

What work is required to do that? The centre of mass will have moved by v upwards, which is the same as if he had taken every part he eats to the top and then removed it – which effectively means he needs to bring the centre of mass of the missing part to the top. Hence we need to find out where the centre of mass was originally located.

Let's assume it was at depth d. In the two-dimensional case this corresponds to the location of the centre of mass of the trapezoid, which is a simple weighted mean of centres of masses of a rectangle and a triangle,

$$d = \frac{\pi a^2 \frac{H-v}{H} v \cdot \frac{v}{2} + \frac{1}{2}\pi a^2 \frac{v}{H} v \cdot \frac{v}{3}}{\pi a^2 \frac{H-v}{H} v + \frac{1}{2}\pi a^2 \frac{v}{H} v} = \frac{\sqrt{2}-1}{3}H. \tag{39.6}$$

The amount of work is equal to the change in potential energy, or

$$W = \frac{V}{2}\rho g d + \frac{V}{2}\rho g v = \frac{4-\sqrt{2}}{6\pi}\frac{V^2}{a^2}\rho g \approx 76\ \mathrm{mJ}. \tag{39.7}$$

40 We are interested in an object moving over distance s with constant acceleration a. Distance grows with time as $s = \frac{1}{2}at^2$ and speed as $v = at$. By combining these two functions we get

$$v(s) = \sqrt{2as}. \tag{40.1}$$

Now we want to determine the average measured speed along the drag strip, or the mean speed, except that it is averaged not over time but over distance. The function $v(s) = \sqrt{2as}$ is increasing between $v(0) = 0$ and $v(s) = \sqrt{2as} = v_{\mathrm{max}}$, and its shape is half of a parabola, $y = \sqrt{x}$. What we need to find is the mean height of this curve.

To do that, we need to know the area under this curve. If we are able to integrate, it is easy, we just evaluate

$$\langle v \rangle_x = \frac{1}{s}\int_0^s \sqrt{2ax}\,\mathrm{d}x = \frac{2\sqrt{2}}{3}\sqrt{as}. \tag{40.2}$$

If we are not, or just do not want to, we can use an old Greek trick, a *quadrature of the parabola*[9]. Let us cut out the area between the line connecting the ends of the curve. The area of the triangle is trivial, and what remains is bounded by a parabola and a line. It is not difficult to show that the area of this shape is equal to $\frac{4}{3}$ of the area of an inscribed triangle with the third vertex in the middle of the interval. After removing this triangle, we are left with two smaller areas, again bounded by a line and a parabola, and this leads to a geometric series with a common ratio $\frac{1}{4}$. The area is naturally the same as in equation 40.2.

[9] Τετραγωνισμὸς παραβολῆς, Archimedes, 3*rd* century BC. Modern version available on Wikipedia.

Now we need to find the actual average speed of Tom's car, or the mean value over time. This is simply the ratio of the total distance and total time[10]

$$\langle v \rangle_t = \frac{s}{\sqrt{\frac{2s}{a}}} = \sqrt{\frac{as}{2}}. \tag{40.3}$$

The ratio of the average speed over distance to average speed over time is therefore

$$\frac{\langle v \rangle_x}{\langle v \rangle_t} = \frac{\frac{2\sqrt{2}}{3}\sqrt{as}}{\sqrt{\frac{as}{2}}} = \frac{4}{3}. \tag{40.4}$$

[10] Note that this definition is consistent with the previous definition of the mean. Average speed is

$$\langle v \rangle_t = \frac{1}{T}\int_0^T at \,\mathrm{d}t = \frac{\frac{1}{2}aT^2}{T} = \frac{s}{T}.$$

Answers

1 0 cm

2 v

3 1 g and 5 g, *or* Earth and Pluto

4 400 ml/s

5 $\frac{826 + 160\sqrt{2}}{343}\ \mathrm{s} \doteq 3.07\ \mathrm{s}$

6 72 s, *also accept* `01:12`.

7 `22:20`

8 100 %, the entire volume of the room.

9 5.47 km

10 2.5 J, *accept results in range* 2.4 J – 2.5 J.

11 $s = \frac{c \cdot t \cdot v}{c - v} = \frac{v^2 t}{c - v} + vt$

12 A–49

13 $s\left(1 + \frac{v}{u}\right)$

14 25.4°

15 3.20 kN, *also accept* 3.14 kN.

16 $\sqrt{8}\ \mathrm{s} \doteq 2.83\ \mathrm{s}$

17 $\frac{1}{3}g$

18 18

19 $\left\{\frac{1}{2}R; \frac{5}{8}R; R\right\}$

20 `N`, `H`, `A`, `B`, `Y`, `I`, *in **that** order.*

21 $56\ \mathrm{mm}^2$

22 7 %

23 $\frac{\cos\alpha}{\sqrt{3\sin^2\alpha+1}} = \frac{\cos\alpha}{\sqrt{4-3\cos^2\alpha}} = \frac{\cot\alpha}{\sqrt{4+\cot^2\alpha}}$

24 74 %

25 10 m, *also accept* 9.8 m.

26 $72\,575\ \mathrm{belch}\cdot\mathrm{wink}^3$

27 $\frac{23255}{7776}\ \mathrm{MPa} \doteq 3.0\ \mathrm{MPa}$

28 0.302 s

29 $-\frac{6\sqrt{3}+3\sqrt{6}+2}{8}\ \mathrm{C} \doteq -2.47\ \mathrm{C}$, *pay attention to the **sign**.*

30 24 Hz

31 1400 %

32 0.26 N

33 33.4°

34 1.51 s, *accept results in range* 1.51 s – 1.52 s.

35 $34\ \mathrm{mm/s^2}$

36 11.8 %

37 $\dfrac{3\pi - 4}{9\pi - 18} mr^2 \doteq 0.528 mr^2$

38 1296 m

39 76 mJ, *accept results in range* 76 mJ – 77 mJ.

40 $\dfrac{4}{3}$

Problems

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